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OF
STATICS AND DYNAMICS

BY

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PREFACE.

IN the following work I have aimed at writing a working text-book on Statics and Dynamics for the use of Junior Students.

The present book, Part I., consists of the Elements of Statics; Part II. will treat of the Elements of Dynamics, and will be shortly published. Both parts will also be bound together and issued as one volume.

Throughout the book will be found a large number of examples; most of them, with the exception of many of those at the end of the Chapter on Friction and the Miscellaneous Examples at the end of the book, are of an easy type.

I have tried to make the book complete as far as it goes; it is suggested, however, that the student should, on a first reading of the subject, omit everything marked with an asterisk.

A few suggestions are subjoined for the guidance of students reading for examinations, where the amount required is less than the whole of the book.

I must express my obligations to my friend Mr H. C. Robson, M.A., Fellow and Lecturer of Sidney Sussex College, Cambridge, for his kindness in reading through the proof-sheets, and for many suggestions that he has made to me.

Any corrections of errors, or hints for improvement will be thankfully received.

S. L. LONEY.

BARNES, S.W.
December, 1890.

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Woolwich Entrance Examination. (Obligatory Papers.) Omit Chapters XII—XIV and all articles marked with an asterisk.

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STATICS.

CHAPTER I.

INTRODUCTION.

1. **A Body** is a portion of matter limited in every direction.

2. **Force** is anything which changes, or tends to change, the state of rest, or uniform motion, of a body.

3. **Rest.** A body is said to be at rest when it does not change its position with respect to surrounding objects.

4. **Statics** is the science which treats of the action of forces on bodies, the forces being so arranged that the bodies are at rest.

The science which treats of the action of force on bodies in motion is called **Dynamics.**

In the more modern system of nomenclature which is gradually gaining general acceptance, the science which treats of the action of force on bodies is called Dynamics, and it has two subdivisions, Statics and Kinetics, treating of the action of forces on bodies which are at rest and in motion respectively.

5. **A Particle** is a portion of matter which is indefinitely small in size.

6. **A Rigid Body** is a body whose parts always preserve an invariable position with respect to one another

In nature no body is perfectly rigid. Every body yields, perhaps only very slightly, if force be applied to it. If a rod, made of wood, have one end firmly fixed and the other end be pulled, the wood stretches slightly; if the rod be made of iron the deformation is very much less.

To simplify our enquiry we shall assume that all the bodies with which we have to deal are perfectly rigid.

7. **Equal Forces.** Two forces are said to be equal when, if they act on a particle in opposite directions, the particle remains at rest.

8. **Mass.** The mass of a body is the quantity of matter in the body. The unit of mass used in England is a pound and is defined to be the mass of a certain piece of platinum kept in the Exchequer Office.

Hence the mass of a body is two, three, four ... lbs., when it contains two, three, four ... times as much matter as the standard lump of platinum.

9. **Weight.** The idea of weight is one with which everyone is familiar. We all know that a certain amount of exertion is required to prevent any body from falling to the ground. The earth attracts every body to itself with a force which, as we shall see in Dynamics, is proportional to the mass of the body.

The force with which the earth attracts any body to itself is called the weight of the body.

10. *Measurement of Force.* We shall choose, as our unit of force in Statics, the weight of one pound. The unit of force is therefore equal to the force which would just support a mass of one pound when hanging freely.

We shall find in Dynamics that the weight of one pound is not quite the same at different points of the earth's surface.

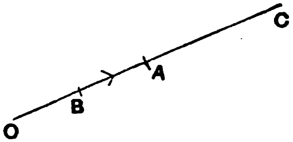
In Statics however we shall not have to compare forces at different points of the earth's surface so that this variation in the weight of a pound is of no practical importance; we shall therefore neglect this variation and assume the weight of a pound to be constant.

11. In practice the expression "weight of one pound" is, in Statics, often shortened into "one pound." The student will therefore understand that "a force of 10 lbs." means "a force equal to the weight of 10 lbs."

12. *Forces represented by straight lines.* A force will be completely known when we know (i) its magnitude, (ii) its direction, and (iii) its point of application, *i.e.* the point of the body at which the force acts.

Hence we can conveniently represent a force by a straight line drawn through its point of application; for a straight line has both magnitude and direction.

Thus suppose a straight line OA represents a force, equal to 10 lbs. weight, acting at a point O. A force of 5 lbs. weight acting in the same direction would be represented by OB, where B bisects the distance OA, whilst a force, equal to 20 lbs. weight, would be represented by OC where OA is produced till AC equals OA.



An arrowhead is often used to denote the direction in which a force acts.

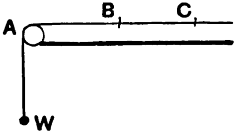
13. *Subdivisions of Force.* There are three different forms under which a force may appear when applied to a mass, viz. as (i) an attraction, (ii) a tension, and (iii) a pressure.

14. *Attraction.* An attraction is a force exerted by one body on another without the intervention of any visible instrument and without the bodies being necessarily in contact. The only example we shall have in this book is the attraction which the earth has for every body; this attraction is (Art. 9) called its weight.

15. *Tension.* If we tie one end of a string to any point of a body and pull at the other end of the string we exert a force on the body; such a force, exerted by means of a string or rod, is called a **tension.**

If the string be light [*i.e.* one whose weight is so small that it may be neglected] the force exerted by the string is the same throughout its length.

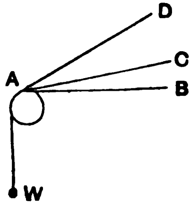
For example, if a weight W be supported by means of a light string passing over the edge of a table it is found



that the same force must be applied to the string whatever be the point, A, B, or C of the string at which the force is applied.

Now the force at A required to support the weight is the same in each case; hence it is clear that the effect at A is the same whatever be the point of the string to which the tension is applied and that the tension of the string is therefore the same throughout its length.

Again if the weight W be supported by a light string passing round a smooth peg A, it is found that the same force must be exerted at the other end of the string whatever be the direction (AB, AC, or AD) in which the string is pulled and that this force is equal to the weight W.



Hence *the tension of a light string passing round a smooth peg is the same throughout its length.*

If two or more strings be knotted together the tensions are not necessarily the same in each string.

The student must carefully notice that the tension of a string is not proportional to its length. It is a common error to suppose that the longer a string the greater is its tension; it is true that we can often apply our force more advantageously if we use a longer piece of string and hence a beginner often assumes that, other things being equal, the longer string has the greater tension.

16. *Pressure.* If one body lean, or be pressed, against another body, each body experiences a force at the point of contact; such a force is called a pressure.

The pressure of one body on a second body is equal and opposite to the pressure of the second body on the first. This statement will be found to be included in Newton's Third Law of Motion.

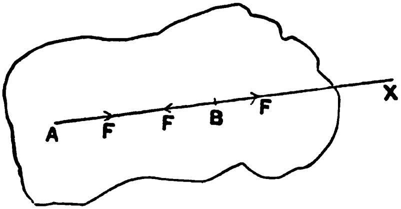
Example. If a ladder lean against a wall the force exerted by the end of the ladder upon the wall is equal and opposite to that exerted by the wall upon the end of the ladder.

17. **Equilibrium.** When two or more forces act upon a body and are so arranged that the body remains at rest, the forces are said to be in equilibrium.

18. *Introduction, or removal, of equal and opposite forces.* We shall assume that if at any point of a rigid body we apply two equal and opposite forces they will have no effect on the equilibrium of the body; similarly that if at any point of a body two equal and opposite forces are acting they may be removed.

19. *Principle of the Transmissibility of Force. If a force act at any point of a rigid body it may be considered to act at any other point in its line of action provided that this latter point be rigidly connected with the body.*

Let a force F act at a point A of a body in a direction AX. Take any point B in AX and at B introduce two



equal and opposite forces, each equal to F, acting in the directions BA and BX; these will have no effect on the equilibrium of the body.

The forces F acting at A in the direction AB, and F at B in the direction BA are equal and opposite; we shall assume that they neutralise one another and hence that they may be removed.

We have thus left the force F at B acting in the direction BX and its effect is the same as that of the original force F at A.

The internal forces in the above body would be different according as the force F is supposed applied at A or B;

of the internal forces, however, we do not treat in the present book.

20. *Smooth bodies.* If we place a piece of smooth polished wood having a plane face upon a table whose top is made as smooth as possible we shall find that if we attempt to move the block along the surface of the table some resistance is experienced. There is always some force, however small, between the wood and the surface of the table.

If the bodies were perfectly smooth there would be no force, parallel to the surface of the table, between the block and the table; the only force between them would be perpendicular to the table.

DEF. When two bodies are perfectly smooth the pressure, or reaction, between them is perpendicular to their common surface at the point of contact.

CHAPTER II.

COMPOSITION AND RESOLUTION OF FORCES.

21. Suppose a flat piece of wood is resting on a smooth table and that it is pulled by means of three strings attached to three of its corners, the forces exerted by the strings being horizontal; if the tensions of the strings be so adjusted that the wood remains at rest it follows that the three forces are in equilibrium.

Hence two of the forces must together exert a force equal and opposite to the third. This force, equal and opposite to the third, is called the resultant of the first two.

22. **Resultant. Def.** *If two or more forces P, Q, S ... act upon a rigid body and if a single force, R, can be found whose effect upon the body is the same as that of the forces P, Q, S ... this single force R is called the resultant of the other forces and the forces P, Q, S ... are called the components of R.*

It follows from the definition that if a force be applied to the body equal and opposite to the force R then the forces acting on the body will balance and the body be in equilibrium; conversely, if the forces acting on a body balance then either of them is equal and opposite to the resultant of the others.

23. *Resultant of forces acting in the same straight line.*

If two forces act on a body in the same direction their resultant is clearly equal to their sum; thus two forces acting in the same direction, equal to 5 and 7 lbs. weight respectively, are equivalent to a force of 12 lbs. weight acting in the same direction as the two forces.

If two forces act on a body in opposite directions their resultant is equal to their difference and acts in the direction of the greater; thus two forces acting in opposite directions and equal to 9 and 4 lbs. weight respectively are equivalent to a force of 5 lbs. weight acting in the direction of the first of the two forces.

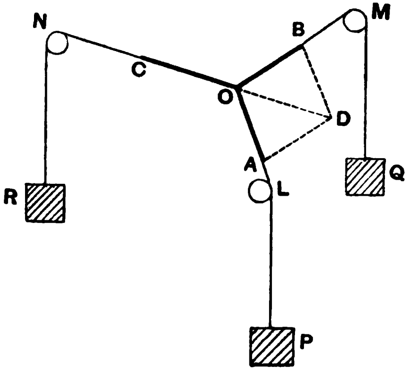
24. When two forces act at a point of a rigid body in different directions their resultant may be obtained by means of the following

Theorem. Parallelogram of Forces. *If two forces acting at a point be represented in magnitude and direction by the two sides of a parallelogram drawn from one of its angular points, their resultant is represented both in magnitude and direction by the diagonal of the parallelogram passing through that angular point.*

In the following article we shall give an experimental proof; the formal proof will be found in Art. 43.

25. **Experimental proof.**

Let L, M, N be three small smooth pegs over which pass light strings supporting masses P lbs., Q lbs., and R lbs. respectively. Let one end of each of these strings be tied together at a point O; then [unless two of the weights are together less than the third] the system will take up some such position as that in the figure. The tension of the string OL is unaltered by passing round the smooth peg L and is therefore equal to the weight of P lbs.; so the tensions in the strings *OM*, *ON* are respectively the weights of *Q* and *R* lbs.



Hence the point *O* is in equilibrium under the action of forces which are equal respectively to the weights of *P*, *Q*, and *R* lbs.

Along *OL*, *OM*, *ON* measure off distances *OA*, *OB*, *OC* proportional to *P*, *Q*, and *R* respectively and complete the parallelogram *OADB*.

Then it will be found that *OD* is exactly equal in magnitude, and opposite in direction, to *OC*.

But the effect of the forces *OA* and *OB* is equal and opposite to that of *OC*. Hence the effect of the force *OD* is exactly the same as that of forces *OA* and *OB*.

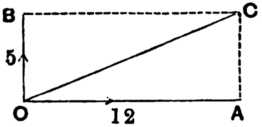
This will be found to be true whatever be the relative magnitudes of *P*, *Q*, and *R*, provided only that one of them is not greater than the sum of the other two.

Hence we conclude that the theorem enunciated is always true.

26. To find the direction and magnitude of the resultant of two forces we have to find the direction and magnitude of the diagonal of a parallelogram of which the two sides represent the forces.

Ex. 1. *Find the resultant of forces equal to* 12 *and* 5 *lbs. weight respectively acting at right angles.*

Let OA and OB represent the forces so that OA is 12 units of length and OB is 5 units of length; complete the rectangle $OACB$.



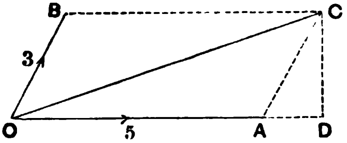
Then $OC^2=OA^2+AC^2=12^2+5^2=169. \quad \therefore OC=13.$

Also $$\tan COA=\frac{AC}{OA}=\tfrac{5}{12}.$$

Hence the resultant is a force equal to 13 lbs. weight making with the first force an angle whose tangent is $\frac{5}{12}$, *i.e.* about 22° 37′.

Ex. 2. *Find the resultant of forces equal to the weights of* 5 *and* 3 *lbs. respectively acting at an angle of* 60°.

Let OA, OB represent the forces, so that OA is 5 units and OB 3 units of length; also let the angle AOB be 60°.



Complete the parallelogram $OACB$ and draw CD perpendicular to OA. Then OC represents the required resultant.

Now $AD=AC\cos CAD=3\cos 60^\circ=\frac{3}{2}; \quad \therefore OD=\frac{13}{2}.$

Also $$DC=AC\sin 60^\circ=3\frac{\sqrt{3}}{2}.$$

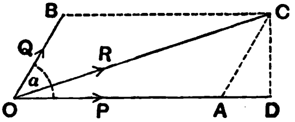
$$\therefore \quad OC=\sqrt{OD^2+DC^2}=\sqrt{\tfrac{169}{4}+\tfrac{27}{4}}=\sqrt{49}=7,$$

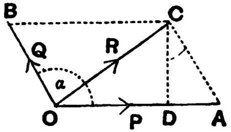
and $$\tan COD = \frac{DC}{OD} = \frac{3\sqrt{3}}{13}$$

Hence the resultant is a force equal to 7 lbs. weight in a direction making with OD an angle whose tangent is $\frac{3\sqrt{3}}{13}$, *i.e.* about 21° 47′.

27. The resultant, R, of two forces P, Q acting at an angle α may be easily obtained by Trigonometry.

For let OA, OB represent the forces P and Q acting at





an angle α. Complete the parallelogram $OACB$ and draw CD perpendicular to OA, produced if necessary.

Let R denote the magnitude of the resultant.

Then $$OD = OA + AD = OA + AC \cos DAC$$
$$= P + Q \cos BOD = P + Q \cos \alpha.$$

[If D fall between O and A as in the second figure we have $OD = OA - DA = OA - AC \cos DAC = P - Q \cos(\pi - \alpha) = P + Q \cos \alpha$.]

Also $$DC = AC \sin DAC = Q \sin \alpha.$$

$$\therefore\ R^2 = OC^2 = OD^2 + CD^2 = (P + Q \cos \alpha)^2 + (Q \sin \alpha)^2$$
$$= P^2 + 2PQ \cos \alpha + Q^2.$$

$$\therefore\ R = \sqrt{P^2 + 2PQ \cos \alpha + Q^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

Also $$\tan COD = \frac{DC}{OD} = \frac{Q \sin \alpha}{P + Q \cos \alpha} \ldots\ldots\ldots\ldots \text{(ii)}.$$

These two equations give the required magnitude and direction of the resultant.

Cor. 1. If the forces be at right angles we have $\alpha = \frac{\pi}{2}$, so that $R = \sqrt{P^2 + Q^2}$, and $\tan COA = \frac{Q}{P}$.

Cor. 2. If the forces be each equal to P we have

$$R = \sqrt{P^2(1 + 2\cos\alpha + 1)} = P\sqrt{2(1 + \cos\alpha)}$$

$$= P\sqrt{2 \cdot 2\cos^2\frac{\alpha}{2}} = 2P\cos\frac{\alpha}{2},$$

and
$$\tan COA = \frac{P\sin\alpha}{P + P\cos\alpha}$$

$$= \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \tan\frac{\alpha}{2},$$

so that the resultant of two equal forces bisects the angle between them.

EXAMPLES. I.

1. In the following seven examples P and Q denote two component forces acting at an angle α and R denotes their resultant.

(i). If $P=24$; $Q=7$; $\alpha=90°$; find R.
(ii). If $P=13$; $R=14$; $\alpha=90°$; find Q.
(iii). If $P=7$; $Q=8$; $\alpha=60°$; find R.
(iv). If $P=5$; $Q=9$; $\alpha=120°$; find R.
(v). If $P=3$; $Q=5$; $R=7$; find α.
(vi). If $P=13$; $Q=14$; $\alpha=\sin^{-1}\frac{12}{13}$; find R.
(vii). If $P=5$; $R=7$; $\alpha=60°$; find Q.

2. Find the greatest and least resultants of two forces whose magnitudes are 12 and 8 lbs. weight respectively.

3. Forces equal respectively to 3, 4, 5, and 6 lbs. weight act on a particle in directions respectively north, south, east, and west; find the direction and magnitude of their resultant.

4. Forces of 84 and 187 lbs. weight act at right angles; find their resultant.

5. Two forces whose magnitudes are P and $P\sqrt{2}$ lbs. weight act on a particle in directions inclined at an angle of 135° to each other; find the magnitude and direction of the resultant.

6. Two forces acting at an angle of 60° have a resultant equal to $2\sqrt{3}$ lbs. weight; if one of the forces be 2 lbs. weight, find the other force.

7. Find the resultant of two forces equal to the weights of 13 and 11 lbs. respectively acting at an angle whose tangent is $\frac{12}{5}$.

8. Find the resultant of two forces equal to the weights of 10 and 9 lbs. respectively acting at an angle whose tangent is $\frac{4}{3}$.

9. Two equal forces act on a particle; find the angle between them when the square of their resultant is equal to three times their product.

10. Find the magnitude of two forces such that, if they act at right angles, their resultant is $\sqrt{10}$ lbs. weight, whilst when they act at an angle of 60° their resultant is $\sqrt{13}$ lbs. weight.

11. Find the angle between two equal forces P when their resultant is (i) equal to P, (2) equal to $\frac{P}{2}$.

12. At what angle do forces equal to $(A+B)$ and $(A-B)$ act so that the resultant may be $\sqrt{A^2+B^2}$?

13. Two given forces act on a particle; find in what direction a third force of given magnitude must act so that the resultant of the three may be as great as possible.

28. Two forces, given in magnitude and direction, have only one resultant; for only one parallelogram can be constructed having two lines OA, OB (Fig. Art. 27) as adjacent sides.

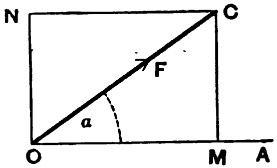
29. A force may be resolved into two components in an infinite number of ways; for an infinite number of parallelograms can be constructed having OC as a diagonal and each of these parallelograms would give a pair of such components.

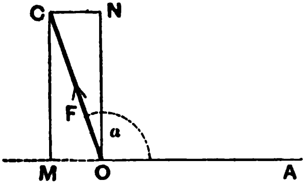
30. The most important case of the resolution of forces occurs when we resolve a force into two components at right angles to one another.

Suppose we wish to resolve a force F, represented by OC, into two components, one of which is in the direction OA and the other is perpendicular to OA.

Draw CM perpendicular to OA and complete the parallelogram $OMCN$. The forces represented by OM and ON

have as their resultant the force OC, so that OM and ON are the required components.





Let the angle AOC be α.

Then $$OM = OC \cos\alpha = F\cos\alpha,$$
and $$ON = MC = OC \sin\alpha = F \sin\alpha.$$

[If the point M lie in OA produced backwards as in the second figure, the component of F in the direction OA

$$= -OM = -OC \cos COM = OC \cos\alpha = F\cos\alpha.$$

Also the component perpendicular to OA

$$= ON = MC = OC \sin COM = F \sin\alpha.]$$

Hence, in each case, the required components are

$$F\cos\alpha \text{ and } F\sin\alpha.$$

Thus a force equal to 10 lbs. weight acting at an angle of 60° with the horizontal is equivalent to $10 \cos 60° \ (= 10 \times \frac{1}{2} = 5$ lbs. weight) in a horizontal direction, and $10 \sin 60° \ (= 10 \times \frac{\sqrt{3}}{2} = 5 \times 1{\cdot}732 = 8{\cdot}66$ lbs. weight) in a vertical direction.

31. **Def.** -The Resolved Part of a given force in a given direction is the component in the given direction which, with a component in a direction perpendicular to the given direction, is equivalent to the given force.

Thus in the previous article the resolved part of the force F in the direction OA is $F\cos\alpha$. Hence

The Resolved Part of a given force in a given direction is obtained by multiplying the given force by the cosine of the angle between the given force and the given direction.

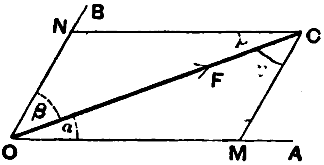
32. A force cannot produce any effect in a direction perpendicular to its own line of action. For (Fig. Art. 30) there is no reason why the force *ON* should have any tendency to make a particle at *O* move in the direction *OA* rather than to make it move in the direction *AO* produced; hence the force *ON* cannot have any tendency to make the particle move in either the direction *OA* or *AO* produced.

For example, if a railway carriage be standing at rest on a railway line it cannot be made to move along the rails by any force which is acting horizontally and in a direction perpendicular to the rails.

33. The resolved part of a given force in a given direction represents the *whole effect* of the force in the given direction. For, (Fig. Art. 30) the force *OC* is completely represented by the forces *ON* and *OM*. But the force *ON* has no effect in the direction *OA*. Hence the whole effect of the force *F* in the direction *OA* is represented by *OM*, *i.e.* by the resolved part of the force in the direction *OA*.

34. A force may be resolved into two components in any two assigned directions.

Let the components of a force *F*, represented by *OC*, in



the directions *OA* and *OB* be required and let the angles *AOC*, *COB* be α, β respectively.

Draw *CM* parallel to *OB* to meet *OA* in *M* and complete the parallelogram *OMCN*.

Then OM, ON are the required components.

Since MC, ON are parallel we have

$$OCM=\beta;\ \text{also}\ OMC=\pi-CMA=\pi-(\alpha+\beta).$$

Since the sides of the triangle OMC are proportional to the sines of the opposite angles, we have

$$\frac{OM}{\sin OCM}=\frac{MC}{\sin MOC}=\frac{OC}{\sin OMC}.$$

$$\therefore \qquad \frac{OM}{\sin\beta}=\frac{MC}{\sin\alpha}=\frac{F}{\sin(\alpha+\beta)}.$$

Hence the required components are

$$F\frac{\sin\beta}{\sin(\alpha+\beta)}\ \text{and}\ F\frac{\sin\alpha}{\sin(\alpha+\beta)}.$$

35. The student must carefully notice that the components of a force in two assigned directions are not the same as the resolved parts of the forces in these directions. For example, the resolved part of F in the direction OA is, by Art. 30, $F\cos\alpha$.

EXAMPLES. II.

1. A force equal to 10 lbs. weight is inclined at an angle of 30° to the horizontal; find its resolved parts in a horizontal and vertical direction respectively.

2. Find the resolved part of a force P in a direction making (i) an angle of 45°, (2) an angle equal to $\cos^{-1}\left(\frac{12}{13}\right)$ with its direction.

3. A truck is at rest on a railway line and is pulled by a horizontal force equal to the weight of 100 lbs. in a direction making an angle of 60° with the direction of the rails; what is the force tending to urge the truck forwards?

4. Resolve a force of 100 lbs. weight into two equal forces acting at an angle of 60°.

5. Resolve a force of 50 lbs. weight into two forces making angles of 60° and 45° with it on opposite sides.

6. Find the components of a force P along two directions making angles of 30° and 45° with P on opposite sides.

7. If a force P be resolved into two forces making angles of 45° and 15° with its direction, shew that the latter force is $P\sqrt{\frac{2}{3}}$.

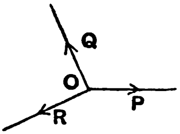
8. Find a horizontal force and a force inclined at an angle of 60° with the vertical whose resultant shall be a given vertical force F.

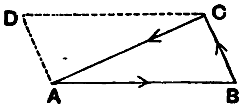
9. If a force be resolved into two component forces and if one component be at right angles to the force and equal to it in magnitude, find the direction and magnitude of the other component.

10. A force equal to the weight of 20 lbs. acting vertically upwards is resolved into two forces, one being horizontal and equal to the weight of 10 lbs.; what is the magnitude and direction of the other force?

36. **Triangle of Forces.** *If three forces, acting at a point, be represented in magnitude and direction by the sides of a triangle, taken in order, they will be in equilibrium.*

Let the forces P, Q, R acting at the point O be represented in magnitude and direction by the sides AB, BC, CA of the triangle ABC; they shall be in equilibrium.





Complete the parallelogram $ABCD$.

The forces represented by BC and AD are the same, since BC and AD are equal and parallel.

Now the resultant of the forces AB, AD, is, by the parallelogram of forces, represented by AC.

Hence the resultant of AB, BC, CA is equal to the resultant of forces AC and CA, and is therefore zero.

Hence the three forces P, Q, R are in equilibrium.

Cor. Since forces represented by AB, BC, and CA

are in equilibrium, and since, when three forces are in equilibrium, each is equal and opposite to the resultant of the other two, it follows that the resultant of *AB* and *BC* is equal and opposite to *CA*, *i.e.* their resultant is represented by *AC*.

Hence the resultant of two forces, acting at a point and represented by the sides *AB*, *BC* of a triangle, is represented by the third side *AC*.

37. In the Triangle of Forces the student must carefully note that the forces must be parallel to the sides of a triangle *taken in order*.

For example, if the first force act in the direction *AB*, the second must act in the direction *BC*, and the third in the direction *CA*; if the second force were in the direction *CB*, instead of *BC*, the forces would not be in equilibrium.

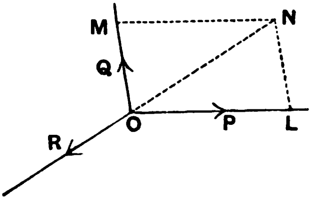
The three forces must also act *at a point;* if the lines of action of the forces were *BC*, *CA*, and *AB* they would not be in equilibrium; for the forces *AB*, *BC* would have a resultant, acting at *B*, equal and parallel to *AC*. The system of forces would then reduce to two equal and parallel forces acting in opposite directions, and, as we shall see in a later chapter, such a pair of forces could not be in equilibrium.

38. The converse of the Triangle of Forces is also true, viz. that *If three forces acting at a point be in equilibrium they can be represented in magnitude and direction by the sides of a triangle taken in order.*

Let the three forces *P*, *Q*, and *R*, acting at a point *O*, be in equilibrium. Measure off lengths *OL* and *OM* along the directions of *P* and *Q* to represent these forces respectively.

Complete the parallelogram *OLMN* and join *ON*.

Since the three forces *P*, *Q*, *R* are in equilibrium each must be equal and opposite to the resultant of the other two. Hence *R* must be equal and opposite to the resultant of *P* and *Q*, and must therefore be represented by *NO*.



Also *LN* is equal and parallel to *OM*.

Hence the three forces *P*, *Q*, and *R* are parallel and proportional to the sides *OL*, *LN*, *NO* of the triangle *OLN*.

Any other triangle, whose sides are parallel to those of the triangle *OLN*, will have its sides proportional to those of *OLN* and therefore proportional to the forces.

Again any triangle, whose sides are respectively perpendicular to those of the triangle *OLN*, will have its sides proportional to the sides of *OLN* and therefore proportional to the forces.

39. The proposition of the last article gives an easy method of determining the relative directions of three forces which are in equilibrium and whose magnitudes are known. We have to construct a triangle whose sides are proportional to the forces, and this, by Euc. I. 22, can always be done unless two of the forces added together are less than the third.

40. **Lami's Theorem.** *If three forces acting on a particle keep it in equilibrium, each is proportional to the sine of the angle between the other two.*

Taking Fig., Art. 38, let the forces P, Q, R be in equilibrium. As before, measure off lengths OL, OM to represent the forces P and Q and complete the parallelogram $OLNM$. Then NO represents R.

Since the sides of the triangle OLN are proportional to the sines of the opposite angles, we have

$$\frac{OL}{\sin LNO} = \frac{LN}{\sin LON} = \frac{NO}{\sin OLN}.$$

But

$$\sin LNO = \sin NOM = \sin(\pi - QOR) = \sin QOR,$$

$$\sin LON = \sin(\pi - LOR) = \sin ROP,$$

and

$$\sin OLN = \sin(\pi - POQ) = \sin POQ.$$

Also

$$LN = OM.$$

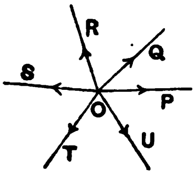
Hence

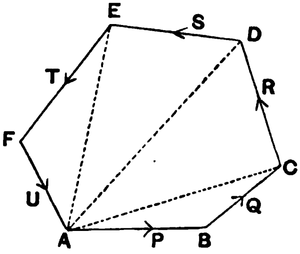
$$\frac{OL}{\sin QOR} = \frac{OM}{\sin ROP} = \frac{NO}{\sin POQ},$$

i.e.

$$\frac{P}{\sin QOR} = \frac{Q}{\sin ROP} = \frac{R}{\sin POQ}.$$

41. **Polygon of Forces.** *If any number of forces, acting on a particle, be represented, in magnitude and direction, by the sides of a polygon, taken in order, the forces shall be in equilibrium.*





Let the sides AB, BC, CD, DE, EF, FA of the polygon

ABCDEF represent the forces acting on a particle *O*. Join *AC*, *AD*, *AE*.

By the corollary to Art. 36, the resultant of forces *AB*, *BC* is represented by *AC*.

Similarly the resultant of forces *AC*, *CD* is represented by *AD*; the resultant of forces *AD*, *DE* by *AE*; and the resultant of forces *AE*, *EF* by *AF*.

Hence the resultant of all the forces is equal to the resultant of *AF* and *FA*, *i.e.* the resultant vanishes.

Hence the forces are in equilibrium.

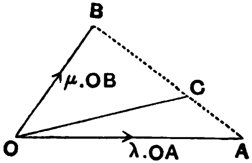
A similar method of proof will apply whatever be the number of forces.

The converse of the Polygon of Forces is not true; for the ratios of the sides of a polygon are not known when the directions of the sides are known. For example, in the above figure, we might take any point A' on *AB* and draw $A'F'$ parallel to *AF* to meet *EF* in F'; the new polygon $A'BCDEF'$ has its sides respectively parallel to those of the polygon *ABCDEF* but the corresponding sides are clearly not proportional.

42. *The resultant of two forces, acting at a point O in directions OA and OB and represented in magnitude by* $\lambda \,.\, OA$ *and* $\mu \,.\, OB$, *is represented by* $(\lambda + \mu) \,.\, OC$, *where C is a point in AB such that* $\lambda \,.\, CA = \mu \,.\, CB$.

For let *C* divide the line *AB* such that

$$\lambda \,.\, CA = \mu \,.\, CB.$$



By the corollary to Art. 36, the force $\lambda \,.\, OA$ is equivalent to forces represented by $\lambda \,.\, OC$ and $\lambda \,.\, CA$.

Also the force $\mu . OB$ is equivalent to forces represented by $\mu . OC$ and $\mu . CB$.

Hence the forces $\lambda . OA$ and $\mu . OB$ are together equivalent to a force $(\lambda + \mu) OC$ together with forces $\lambda . CA$ and $\mu . CB$.

But, by construction, these two latter forces are equal and opposite and therefore are in equilibrium.

Hence the resultant is $(\lambda + \mu) . OC$.

Cor. The resultant of forces represented by OA and OB is $2OC$ where C is the middle point of AB.

This is also clear from the fact that OC is half the diagonal OD of the parallelogram of which OA, OB are adjacent sides.

EXAMPLES. III.

1. Three forces acting at a point are in equilibrium; if they make angles of 120° with one another, shew that they are equal.

If the angles are 60°, 150°, and 150°, in what proportions are the forces?

2. Three forces acting on a particle are in equilibrium; the angle between the first and second is 90^0 and between the second and third is 120^0; find the ratios of the forces.

3. Forces equal to $7P$, $5P$, and $8P$ acting on a particle are in equilibrium; find the angle between the latter pair of forces.

4. Forces equal to $5P$, $12P$, $13P$ acting on a particle are in equilibrium; find the angles between their directions.

5. Construct geometrically the directions of two forces $2P$ and $3P$ which make equilibrium with a force of $4P$ whose direction is given.

6. The sides AB, AC of a triangle ABC are bisected in D and E; shew that the resultant of forces represented by BE and DC is represented in magnitude and direction by $\frac{3}{2}BC$.

7. P is a particle acted on by forces $\lambda . AP$ and $\lambda . PB$ where A and B are two fixed points; shew that their resultant is constant in magnitude and direction wherever the point P may be.

8. $ABCD$ is a parallelogram; a particle P is attracted towards A and C by forces which are proportional to PA and PC respectively and repelled from B and D by forces proportional to PB and PD; shew that P is in equilibrium wherever it is situated.

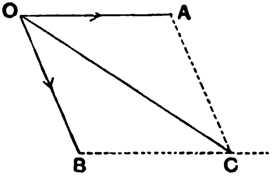
*43. Formal proof of the Parallelogram of Forces.

The proof is divided into two portions, (I) as regards the direction, (II) as regards the magnitude of the resultant.

I. DIRECTION.

(α) Equal Forces.

Let the forces be equal and represented by OA and OB.



Complete the parallelogram $OACB$, and join OC. Then OC bisects the angle AOB.

Since the forces are equal it is clear that the resultant must bisect the angle between them; for there is no reason to shew why the resultant should lie on one side of OC which would not equally hold to shew that the resultant should lie on the other side of OC. Hence, as far as regards direction, we may assume the truth of the theorem for equal forces.

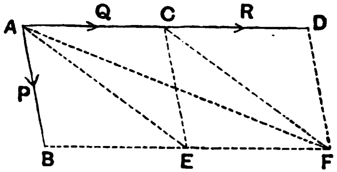
(β) Commensurable Forces.

Lemma. *If the theorem be true, as far as regards direction, for a pair of forces P and Q, and also for a pair of forces P and R acting at the same angle, to shew that it is true for the pair of forces P and $(Q+R)$.*

Let the forces act at a point A of a rigid body, and let AB be the direction of P, and ACD that of Q and R.

Let AB, AC represent the forces P and Q in magnitude.

Since, by the principle of The Transmissibility of Force, the force R may be supposed to act at any point in its line of action, let it act at C and be represented by CD.



Complete the parallelograms $ABEC$, $ABFD$.

The resultant of P and Q is, by supposition, equal to some force T acting in the direction AE; let them be replaced by this resultant and let its point of application be removed to E.

This force T, acting at E, may now be replaced by forces, equal to P and Q, acting in the directions CE and EF respectively.

Let their points of application be removed to C and F.

Again, by the supposition, the resultant of P and R, acting at C, is equivalent to some force acting in the direction CF; let them be replaced by their resultant and let its point of application be removed to F.

All the forces have now been applied at F without altering their combined effect; hence F must be a point on the line of action of their resultant; therefore AF is the direction of the required resultant.

Hence the Lemma is proved.

Application of the lemma.

By (a) we know that the theorem is true for forces which are each equal to S.

Hence, by the lemma, putting P, Q, and R each equal to S, we see that the theorem is true for forces S and $2S$. Again, by the lemma, since the theorem is true for forces (S, S) and (S, $2S$) we see that it is true for forces (S, $3S$). Similarly for forces (S, $4S$) and so on.

Continuing in this way we see that it is true for forces S and mS, where m is any positive integer.

Again, from the lemma, putting P equal to mS, and Q and R both equal to S, the theorem is true for forces mS and $2S$.

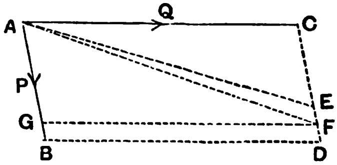
Again, putting P equal to mS, Q to $2S$, and R to S, the theorem is true for forces mS and $3S$.

Proceeding in this way we see that the theorem is true for forces mS and nS, where m, n are positive integers.

Also any two commensurable forces can be represented by mS and nS.

(γ) **Incommensurable Forces.**

Let P and Q be incommensurable forces, and let AB and AC represent them.



Complete the parallelogram $ABDC$.

If the resultant of P and Q be not in the line AD let it act in the line AE meeting CD in E.

Divide AC into any number of equal parts x, *each less than* ED, and from CD cut off successively portions, each equal to x. The last point of subdivision F must fall between E and D, *since* x *is less than* ED.

Draw FG parallel to CA to meet AB in G, and join AF.

The lines AC and AG represent commensurable forces, and therefore their resultant is, by (β), in the direction AF.

Hence the resultant of forces AC and AB must lie within the angle BAF. But this resultant acts in the direction AE, which is without the angle BAE.

But this is absurd.

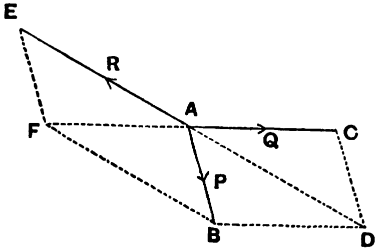
Hence AE cannot be the direction of the resultant.

In a similar manner it can be shewn that no other line, except AD, can be the direction of the resultant.

Hence AD is the direction of the resultant.

II. MAGNITUDE.

As before let AB, AC represent the forces P and Q. Complete the parallelogram $ABDC$.



Take a force R, represented both in magnitude and direction by AE, to balance the resultant of P and Q.

Then, by the first part of the proof, *AE is in the same straight line with AD.* AE shall also be equal to AD.

Complete the parallelogram $AEFB$.

Since the three forces P, Q, and R are in equilibrium, each of them is equal and opposite to the resultant of the other two.

Now the resultant of P and R is in the direction AF; hence AC, the direction of Q, is in the same straight line with AF.

Therefore $ADBF$ is a parallelogram, and hence AD equals BF.

But, since $AEFB$ is a parallelogram, BF equals AE.

Therefore AD equals AE, and hence AD is equal, in magnitude as well as direction, to the resultant of P and Q.

EXAMPLES. IV.

1. Two forces act at an angle of 120°. The greater is represented by 80 and the resultant is at right angles to the less. Find the latter.

2. If one of two forces be double the other and the resultant be equal to the greater force find the cosine of the angle between the forces.

3. Two forces acting on a particle are at right angles and are balanced by a third force making an angle of 150° with one of them. The greatest force being 3 lbs. weight what must be the values of the other two?

4. The resultant of two forces acting at an angle equal to $\frac{5}{3}$ds of a right angle is perpendicular to the smaller component. The greater being equal to 30 lbs. weight, find the other component and the resultant.

5. The magnitudes of two forces are as 3 : 5, and the direction of the resultant is at right angles to that of the smaller force; compare the magnitudes of the larger force and of the resultant.

6. The sum of two forces is 18, and the resultant, whose direction is perpendicular to the lesser of the two forces, is 12; find the magnitude of the forces.

7. If two forces P and Q act at such an angle that $R=P$, shew that, if P be doubled, the new resultant is at right angles to Q.

8. The resultant of two forces P and Q is equal to $\sqrt{3}Q$ and makes an angle of 30° with the direction of P; shew that P is either equal to, or is double of, Q.

9. Two forces equal to $2P$ and P respectively act on a particle; if the first be doubled and the second increased by 12 lbs. weight the direction of the resultant is unaltered; find the value of P.

10. The resultant of two forces P, Q acting at an angle θ is equal to $(2m+1)\sqrt{P^2+Q^2}$; when they act at an angle $\frac{\pi}{2}-\theta$, the resultant is $(2m-1)\sqrt{P^2+Q^2}$; prove that

$$\tan\theta=\frac{m-1}{m+1}.$$

11. The resultant of forces P and Q is R; if Q be doubled R is doubled, whilst, if Q be reversed, R is again doubled; shew that

$$P:Q:R::\sqrt{2}:\sqrt{3}:\sqrt{2}.$$

12. If the resultant, R, of two forces, P and Q, inclined to one another at any given angle, make an angle θ with the direction of P, shew that the resultant of forces $(P+R)$ and Q, acting at the same given angle, will make an angle $\frac{\theta}{2}$ with the direction of $(P+R)$.

13. Three equal forces acting at a point are in equilibrium. If one of them be turned about its point of application through a given angle, find by a simple construction the resultant of the three, and, if the inclination of the force continue to alter, shew that the inclination of the resultant alters by half the amount.

14. Decompose a force, whose magnitude and line of action are given, into two equal forces passing through two given points, giving a geometrical construction, (1) when the two points are on the same side of the force, (2) when they are on opposite sides.

15. Two given forces act at two given points of a body; if they are turned round those points in the same direction through any two equal angles, shew that their resultant will always pass through a fixed point.

16. A, B, C are three fixed points, and P is a point such that the resultant of forces PA, PB always passes through C; shew that the locus of P is a straight line.

17. A given force acting at a given point in a given direction is resolved into two components. If for all directions of the components one remains of invariable magnitude, shew that the extremity of the line representing the other lies on a definite circle.

18. Shew that the system of forces represented by the lines joining any point to the angular points of a triangle is equivalent to the system represented by straight lines drawn from the same point to the middle points of the sides of the triangle.

19. Find a point within a quadrilateral such that, if it be acted on by forces represented by the lines joining it to the angular points of the quadrilateral, it will be in equilibrium.

20. Four forces act along and are proportional to the sides of the quadrilateral $ABCD$; three act in the directions AB, BC, CD and the fourth acts from A to D; find the magnitude and direction of their resultant, and determine the point in which it meets CD.

21. The sides AB, BC, CD, DA of a quadrilateral $ABCD$ are bisected in E, F, G, H respectively; shew that if two forces parallel and equal to AB and DC act on a particle then the resultant is parallel to HF and equal to $2 . HF$.

22. The sides AB, BC, CD, DA of a quadrilateral $ABCD$ are bisected at E, F, G, H respectively. Shew that the resultant of the forces acting at a point which are represented in magnitude and direction by EG and HF is represented in magnitude and direction by AC.

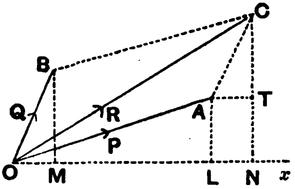
23. From a point, P, within a circle whose centre is fixed, straight lines PA_1, PA_2, PA_3 and PA_4 are drawn to meet the circumference, all being equally inclined to the radius through P; shew that, if these lines represent forces radiating from P, their resultant is independent of the magnitude of the radius of the circle.

CHAPTER III.

COMPOSITION AND RESOLUTION OF FORCES (*continued*).

44. *The sum of the resolved parts of two forces in a given direction is equal to the resolved part of their resultant in the same direction.*

Let OA, OB represent the two forces P and Q, and OC their resultant R, so that $OACB$ is a parallelogram.



Let Ox be the given direction; draw AL BM, CN perpendicular to Ox and AT perpendicular to CN.

The sides of the two triangles OBM, ACT are respectively parallel, and OB is equal to AC in magnitude;

$$\therefore\ OM = AT = LN.$$

Hence $\quad ON = OL + LN = OL + OM.$

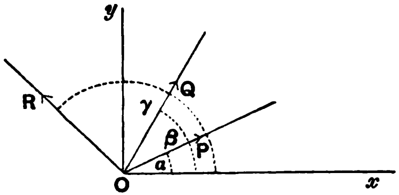
But OL, OM, ON represent respectively the resolved parts of P, Q, and R in the direction Ox.

Hence the theorem is proved.

The theorem may easily be extended to the resultant of any number of forces acting at a point.

45. *To find the resultant of any number of forces in one plane acting upon a particle.*

Let the forces P, Q, R ... act upon a particle at O.



Through O draw a fixed line Ox and a line Oy at right angles to Ox.

Let the forces P, Q, R, ... make angles α, β, γ ... with Ox.

The components of the force P in the directions Ox and Oy are, by Art. 30, $P\cos\alpha$ and $P\sin\alpha$ respectively; similarly, the components of Q are $Q\cos\beta$ and $Q\sin\beta$; similarly for the other forces.

Hence the forces are equivalent to a component,

$$P\cos\alpha + Q\cos\beta + R\cos\gamma \ldots \text{ along } Ox,$$

and a component,

$$P\sin\alpha + Q\sin\beta + R\sin\gamma \ldots \text{ along } Oy.$$

Let these components be X and Y respectively, and let F be their resultant inclined at an angle θ to Ox.

Since F is equivalent to $F\cos\theta$ along Ox, and $F\sin\theta$ along Oy, we have, by the previous article,

$$F\cos\theta = X \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$F\sin\theta = Y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence, by squaring and adding,

$$F^2 = X^2 + Y^2.$$

Also, by division, $\tan\theta = \dfrac{Y}{X}$.

These two equations give F and θ, *i.e.*, the magnitude and direction of the required resultant.

46. The resultant of a system of forces acting at a point may also be obtained by means of the Polygon of Forces. For, (Fig. Art. 41), forces acting at a point O and represented in magnitude and direction by the sides of the polygon $ABCDEF$ are in equilibrium. Hence the resultant of forces represented by AB, BC, CD, DE, EF must be equal and opposite to the remaining force FA, *i.e.*, the resultant must be represented by AF.

It follows that the resultant of forces P, Q, R, S, T acting on a particle may be obtained thus; take a point A and draw AB parallel and proportional to P, and in succession BC, CD, DE, EF parallel and proportional respectively to Q, R, S, and T; the required resultant will be represented in magnitude and direction by the line AF.

The same construction would clearly apply for any number of forces.

47. Ex. 1. *A particle is acted upon by three forces, in one plane, equal to 2, 2 $\sqrt{2}$, and 1 lbs. weight respectively; the first is horizontal, the second acts at 45° to the horizon, and the third is vertical; find their resultant.*

Here $$X = 2 + 2\sqrt{2}\cos 45^\circ + 0 = 2 + 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 4,$$

$$Y = 0 + 2\sqrt{2}\sin 45^\circ + 1 = 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} + 1 = 3.$$

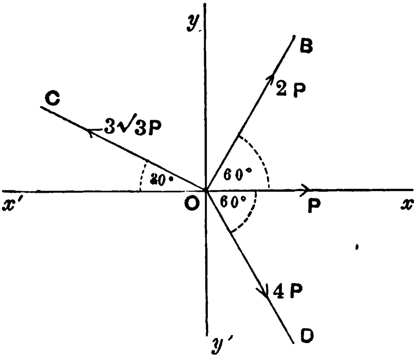
Hence $$F\cos\theta = 4;\ F\sin\theta = 3;$$

$$\therefore\ F = \sqrt{4^2 + 3^2} = 5, \text{ and } \tan\theta = \tfrac{3}{4}.$$

The resultant is therefore a force equal to 5 lbs. weight acting at an angle with the horizontal whose tangent is $\frac{3}{4}$, *i.e.* 36° 52′.

Ex. 2. *A particle is acted upon by forces represented by P, $2P$, $3\sqrt{3}P$, and $4P$; the angles between the first and second, the second and third, and the third and fourth are 60°, 90° and 150° respectively. Shew that the resultant is a force P in a direction inclined at an angle of 120° to that of the first force.*

In this example it will be a simplification if we take the fixed



line Ox to coincide with the direction of the first force P; let xOx' and yOy' be the two fixed lines at right angles.

The second, third, and fourth forces are respectively in the first, second, and fourth quadrants, and we have clearly

$$BOx=60^\circ;\quad COx'=30^\circ;\quad \text{and}\quad DOx=60^\circ.$$

The first force has no component along Oy.

The second force is equivalent to components $2P\cos 60^\circ$ and $2P\sin 60^\circ$ along Ox and Oy respectively.

The third force is equivalent to forces

$$3\sqrt{3}P\cos 30^\circ \quad \text{and} \quad 3\sqrt{3}P\sin 30^\circ$$

along Ox' and Oy respectively, *i.e.* to forces $-3\sqrt{3}P\cos 30^\circ$ and $3\sqrt{3}P\sin 30^\circ$ along Ox and Oy.

So the fourth force is equivalent to $4P\cos 60^\circ$ and $4P\sin 60^\circ$ along Ox and Oy', *i.e.* to $4P\cos 60^\circ$ and $-4P\sin 60^\circ$ along Ox and Oy.

Hence $$X=P+2P\cos 60^\circ-3\sqrt{3}P\cos 30^\circ+4P\cos 60^\circ$$

$$=P+P-\frac{9P}{2}+2P=-\frac{P}{2},$$

and $$Y=O+2P\sin 60^\circ+3\sqrt{3}P\sin 30^\circ-4P\sin 60^\circ$$

$$=P\sqrt{3}+\frac{3\sqrt{3}}{2}P-4P\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}P.$$

Hence, if F be the resultant at an angle θ with Ox, we have

$$F=\sqrt{X^2+Y^2}=P,$$

and $$\tan\theta = \frac{Y}{X} = -\sqrt{3} = \tan 120°,$$

so that the resultant is a force P at an angle of 120° with the first force.

EXAMPLES. V.

1. Forces of 1, 2, and $\sqrt{3}$ lbs. weight act at a point A in directions AP, AQ, and AR, the angle PAQ being 60° and PAR a right angle; find their resultant.

2. A particle is acted on by forces of 5 and 3 lbs. weight which are at right angles and by a force of 4 lbs. weight bisecting the angle between them; find the force that will keep it at rest.

3. Three equal forces, P, diverge from a point, the middle one being inclined at an angle of 60° to each of the others. Find the resultant of the three.

4. Three forces $5P$, $10P$, and $13P$ act in one plane on a particle, the angle between any two of their directions being 120°. Find the magnitude and direction of their resultant.

5. Forces $2P$, $3P$, and $4P$ act at a point in directions parallel to the sides of an equilateral triangle taken in order; find the magnitude and line of action of the resultant.

6. Forces P_1, P_2, P_3, P_4 act on a particle O at the centre of a square $ABCD$; P_1 and P_2 act along the diagonals OA and OB, and P_3 and P_4 perpendicular to the sides AB and BC. If
$$P_1 : P_2 : P_3 : P_4 :: 4 : 6 : 5 : 1,$$
find the magnitude and direction of their resultant.

7. $ABCD$ is a square; forces of 1 lb. wt., 6 lbs. wt., and 9 lbs. wt. act in the directions AB, AC, and AD respectively; find the magnitude of their resultant correct to two places of decimals.

8. Five forces, acting at a point, are in equilibrium; four of them, whose respective magnitudes are 4, 4, 1, and 3 lbs. weight make, in succession, angles of 60° with one another. Find the magnitude of the fifth force.

9. Four equal forces P, Q, R, S act on a particle in one plane; the angles between P, Q, between Q, R, and between R, S are all equal and that between P and S is 108°. Find their resultant.

10. Forces of 2, $\sqrt{3}$, 5, $\sqrt{3}$, and 2 lbs. wt. respectively act at one of the angular points of a regular hexagon towards the five other angular points; find the direction and magnitude of the resultant.

11. Forces of 2, 3, 4, 5, 6 lbs. wt. respectively act at an angular point of a regular hexagon towards the other angular points taken in order; find their resultant.

12. Shew that the resultant of forces equal to 7, 1, 1, and 3 lbs. wt. respectively acting at an angular point of a regular pentagon towards the other angular points, taken in order, is $\sqrt{71}$ lbs. wt.

13. Equal forces P act on an angular point of an octagon towards each of the other angular points; find their resultant.

14. Using trigonometrical tables, find the magnitude (to 2 places of decimals) and the direction (to the nearest minute) of the resultant of

(1) three forces equal to 11, 7, and 8 lbs. weight, making angles of 18° 18′, 74° 50′, and 130° 20′ with a fixed line;

(2) four forces equal to 4, 3, 2, and 1 lb. weight, making angles of 20°, 40°, 60°, and 80° with a fixed line;

(3) four forces equal to 8, 12, 15, and 20 lbs. weight, making angles of 30°, 70°, 120° 15′, and 155° with a fixed line.

48. *To find the conditions of equilibrium of any number of forces acting upon a particle.*

Let the forces act upon a particle O as in Art. 45.

If the forces balance one another the resultant must vanish, *i.e.* F must be zero.

Hence $$X^2 + Y^2 = 0.$$

Now the sum of the squares of two real quantities cannot be zero unless each quantity is separately zero;

$$\therefore X = 0, \text{ and } Y = 0.$$

Hence the forces acting on a particle are in equilibrium if the algebraic sum of their resolved parts in two directions at right angles are separately zero.

Conversely, if the sum of their resolved parts in two directions at right angles separately vanish, the forces are in equilibrium.

For, in this case, both X and Y are zero, and therefore F is zero also.

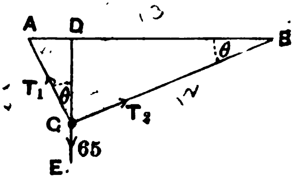
Hence, since the resultant of the forces vanishes, the forces are in equilibrium.

49. When there are only three forces acting on a particle the conditions of equilibrium are often most easily found by applying Lami's Theorem (Art. 40).

50. Ex. 1. *A body of 65 lbs. weight is suspended by two strings of lengths 5 and 12 feet attached to two points in the same horizontal line whose distance apart is 13 feet; find the tensions of the strings.*

Let AC, BC be the two strings, so that

$$AC=5 \text{ ft.}, \; BC=12 \text{ ft.}, \text{ and } AB=13 \text{ ft.}$$



Since $13^2=12^2+5^2$, the angle ACB is a right angle.

Let the direction CE of the weight be produced to meet AB in D; also let the angle CBA be θ, so that ACD is θ also.

Let T_1, T_2 be the tensions of the strings. By Lami's theorem we have

$$\frac{T_1}{\sin ECB}=\frac{T_2}{\sin ECA}=\frac{65}{\sin ACB};$$

$$\therefore \; \frac{T_1}{\sin BCD}=\frac{T_2}{\sin\theta}=\frac{65}{\sin 90^\circ};$$

$$\therefore \; T_1=65\cos\theta, \text{ and } T_2=65\sin\theta.$$

But $$\cos\theta=\frac{BC}{BA}=\frac{12}{13}, \text{ and } \sin\theta=\frac{AC}{AB}=\frac{5}{13};$$

$$\therefore \; T_1=60, \text{ and } T_2=25 \text{ lbs. wt.}$$

Otherwise thus; The triangle ACB has its sides respectively perpendicular to the forces T_1, T_2, and 65;

$$\therefore \; \frac{T_1}{BC}=\frac{T_2}{CA}=\frac{65}{AB};$$

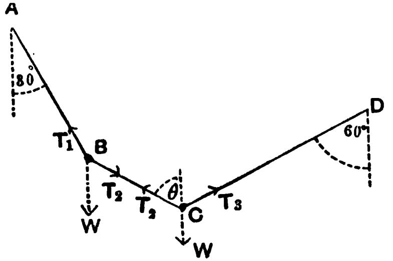
$$\therefore \; T_1=65\frac{BC}{AB}=60, \text{ and } T_2=65\frac{AC}{AB}=25.$$

Ex. 2. *A string ABCD, attached to two fixed points A and D, has two equal weights, W, knotted to it at B and C and rests with the portions AB and CD inclined at angles of 30° and 60° respectively to the vertical. Find the tensions of the portions of the string and the inclination of BC to the vertical.*

Let the tensions in the strings be T_1, T_2, T_3 respectively and let BC be inclined at an angle θ to the vertical.

[N.B. The string BC pulls B towards C and pulls C towards B, the tension being the same throughout its length.]

Since B is in equilibrium the vertical components and the horizontal components of the forces acting on it must both vanish (Art. 48).



Hence $$T_1 \cos 30° - T_2 \cos\theta = W \quad\text{.....................(1)},$$

and $$T_1 \sin 30° - T_2 \sin\theta = 0 \quad\text{......................(2)}.$$

Similarly, since C is in equilibrium,

$$T_3 \cos 60° + T_2 \cos\theta = W \quad\text{.......................(3)},$$

and $$T_3 \sin 60° - T_2 \sin\theta = 0 \quad\text{.........................(4)}.$$

From (1) and (2), substituting for T_1, we have

$$W = T_2 [\cot 30° \sin\theta - \cos\theta] = T_2 [\sqrt{3} \sin\theta - \cos\theta] \quad\text{......(5)}.$$

So from (3) and (4), substituting for T_3, we have

$$W = T_2 [\cot 60° \sin\theta + \cos\theta] = T_2 \left[\frac{1}{\sqrt{3}} \sin\theta + \cos\theta\right] \quad\text{......(6)};$$

therefore from (5) and (6),

$$\sqrt{3} \sin\theta - \cos\theta = \frac{1}{\sqrt{3}} \sin\theta + \cos\theta;$$

$$\therefore \quad 2 \sin\theta = 2\sqrt{3} \cos\theta;$$

$$\therefore \quad \tan\theta = \sqrt{3}, \text{ and hence } \theta = 60°.$$

Substituting this value in (5) we have

$$W = T_2 \left[\sqrt{3} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2}\right] = T_2.$$

Hence from (2) we have

$$T_1 = T_2 \frac{\sin 60°}{\sin 30°} = W \cdot \sqrt{3},$$

and from (4) $$T_3 = T_2 \frac{\sin\theta}{\sin 60°} = T_2 = W.$$

Hence the inclination of BC to the vertical is 60°, and the tensions of the portions AB, BC, CD are $W\sqrt{3}$, W, and W respectively.

EXAMPLES. VI.

1. Two men carry a weight W between them by means of two ropes fixed to the weight; one rope is inclined at 45° to the vertical and the other at 30°; find the tension of each rope.

2. A body, of weight 2 lbs., is fastened to a fixed point by means of a string of length 25 inches; it is acted on by a horizontal force F and rests at a distance of 20 inches from the vertical line through the fixed point; find the value of F and the tension of the string.

3. A body, of 130 lbs. wt., is suspended from a horizontal beam by strings, whose lengths are respectively 1 ft. 4 ins. and 5 ft. 3 ins., the strings being fastened to the beam at two points 5 ft. 5 ins. apart. What are the tensions of the strings?

4. A body, of 70 lbs. wt., is suspended by strings, whose lengths are 6 and 8 feet respectively, from two points in a horizontal line whose distance apart is 10 feet; find the tensions of the strings.

5. A weight of 60 lbs. is suspended by two strings of lengths 9 and 12 feet respectively, the other ends of the strings being attached to two points in a horizontal line at a distance of 15 feet apart; find the tensions of the strings.

6. A string suspended from a ceiling supports three bodies, each of mass four lbs., one at its lowest point and each of the others at equal distances from its extremities; find the tensions of the parts into which the string is divided.

7. Two equal masses, of weight W, are attached to the extremities of a thin string which passes over 3 tacks in a wall arranged in the form of an isosceles triangle, with the base horizontal and with a vertical angle of 120°; find the pressure on each tack.

8. A stream is 96 feet wide and a boat is dragged down the middle of the stream by two men on opposite banks, each of whom pulls with a force equal to 100 lbs. wt.; if the ropes be attached to the same point of the boat and each be of length 60 feet, find the resultant pressure on the boat.

9. A string passing over two smooth parallel bars in the same horizontal plane has two equal weights fastened to its ends and another equal weight is fastened to a point of the string between the bars; find the position of equilibrium of the system and the pressure upon each bar.

10. A string is tied to two points in a horizontal plane; a ring of weight 27 lbs. can slide freely along the string and is pulled by a horizontal force equal to the weight of P lbs. If in the position of equilibrium the portions of the string be inclined at angles of 45° and 75° to the vertical, find the value of P.

11. Two weightless rings slide on a smooth vertical circle and through the rings passes a string which carries weights at the two ends and at a point between the rings. If equilibrium exist when the rings are at points distant 30° from the highest point of the circle, find the relation between the three weights.

12. Two masses, each equal to 112 lbs., are joined by a string which passes over two small smooth pegs, *A* and *B*, in the same horizontal plane; if a mass of 5 lbs. be attached to the string halfway between *A* and *B*, find in inches the depth to which it will descend below the level of *AB*, supposing *AB* to be 10 feet.

What would happen if the small mass were attached to any other point of the string?

13. A body, of 10 lbs. wt., is suspended by two strings, 7 and 24 inches long, their other ends being fastened to the extremities of a rod of length 25 inches. If the rod be so held that the body hangs immediately below its middle point, find the tensions of the string.

14. A heavy chain has weights of 10 and 16 lbs. attached to its ends and hangs in equilibrium over a smooth pulley; if the greatest tension of the chain be 20 lbs. wt., find the weight of the chain.

15. A heavy chain of length 8 ft. 9 ins. and weighing 15 lbs. has a weight of 7 lbs. attached to one end and is in equilibrium hanging over a smooth peg. What length of the chain is on each side?

16. A body is free to slide on a smooth vertical circular wire and is connected by a string, equal in length to the radius of the circle, to the highest point of the circle; find the tension of the string and the pressure on the circle.

17. A uniform plane lamina in the form of a rhombus, one of whose angles is 120°, is supported by two forces applied at the centre in the directions of the diagonals so that one side of the rhombus is horizontal; shew that, if *P* and *Q* be the forces and *P* be the greater, then

$$P^2=3Q^2.$$

18. The ends of a driving rein are passed through two smooth rings which are fastened, one to each side of the bit. They are then doubled back and tied to fixed points in the headpiece one on each side of the horse's head. Find the pressure produced by the bit on the horse's tongue by a given pull *P* of the driver.

19. Three equal strings, of no sensible weight, are knotted together to form an equilateral triangle *ABC* and a weight *W* is suspended from *A*. If the triangle and weight be supported, with *BC* horizontal, by means of two strings at *B* and *C*, each at the angle of 135° with *BC*, shew that the tension in *BC* is

$$\frac{W}{6}(3-\sqrt{3}).$$

20. Three weightless strings AC, BC, and AB are knotted together to form an isosceles triangle whose vertex is C. If a weight W be suspended from C and the whole be supported, with AB horizontal, by two forces bisecting the angles at A and B, find the tension of the string AB.

21. A weightless string is suspended from two points not in the same horizontal line and passes through a small smooth heavy ring which is free to slide on the string; find the position of equilibrium of the ring.

If the ring, instead of being free to move on the string, be tied to a given point of it, find equations to give the ratio of the tensions of the two portions of the string.

22. Four pegs are fixed in a wall at the four highest points of a regular hexagon (the two lowest points of the hexagon being in a horizontal straight line) and over these is thrown a loop supporting a weight W; the loop is of such a length that the angles formed by it at the lowest pegs are right angles. Find the tension of the string and the pressures on the pegs.

23. Explain how a vessel is enabled to sail in a direction nearly opposite to that of the wind.

Shew also that the sails of the vessel should be set so as to bisect the angle between the keel and the apparent direction of the wind in order that the force to urge the vessel forward may be as great as possible.

24. Explain how the force of the current may be used to urge a ferry-boat across the river, assuming that the centre of the boat is attached by a long rope to a fixed point in the middle of the stream.

CHAPTER IV.

PARALLEL FORCES.

51. IN Chapters II. and III. we have shewn how to find the resultant of forces which meet in a point. In the present chapter we shall consider the composition of parallel forces.

In the ordinary statical problems of every-day life parallel forces are of constant occurrence.

52. **Def.** Two parallel forces are said to be **like** when they act in the same direction; when they act in opposite parallel directions they are said to be **unlike.**

53. *To find the resultant of two parallel forces acting upon a rigid body.*

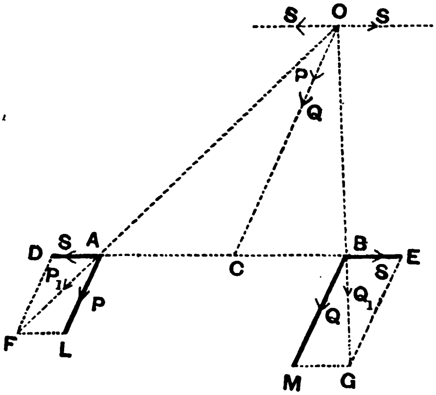
Case I. *Let the forces be* **like.**

Let P, Q be the forces acting at points A and B of the body, and let them be represented by the lines AL and BM.

Join AB and at A and B apply two equal and opposite forces each equal to S and acting in the directions BA and AB respectively. Let these forces be represented by AD and BE. These two forces balance one another and have no effect upon the equilibrium of the body.

Complete the parallelograms $ALFD$ and $BMGE$; let the diagonals FA and GB be produced to meet in O. Draw OC parallel to AL or BM to meet AB in C.

The forces P and S at A have a resultant P_1, represented by AF. Let its point of application be removed to O.



So the forces Q and S at B have a resultant Q_1 represented by BG. Let its point of application be transferred to O.

The force P_1 at O may be resolved into two forces, S parallel to AD, and P in the direction OC.

So the force Q_1 at O may be resolved into two forces, S parallel to BE, and Q in the direction OC.

Also these two forces S acting at O are in equilibrium.

Hence the original forces P and Q are equivalent to a force $(P+Q)$ acting along OC, *i.e.* acting at C parallel to the original directions of P and Q.

To determine the position of the point C. The triangle OCA is, by construction, similar to the triangle ALF;

$$\therefore \frac{OC}{CA} = \frac{AL}{LF} = \frac{P}{S} \quad \text{..................} (1).$$

So, since the triangles *OCB*, *BMG* are similar, we have

$$\frac{OC}{CB} = \frac{BM}{MG} = \frac{Q}{S} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

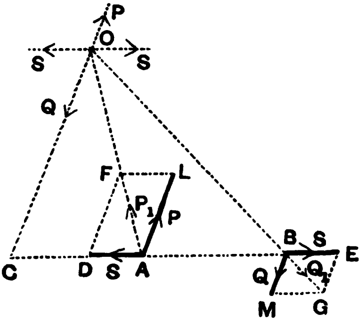
Hence, from (1) and (2), by division,

$$\frac{CA}{CB} = \frac{Q}{P},$$

i.e. *C* divides the line *AB* *internally* in the inverse ratio of the forces.

Case II. *Let the forces be* **unlike.**

Let *P*, *Q* be the forces (*P* being the greater) acting at



points *A*, *B* of the body, and let them be represented by the lines *AL* and *BM*.

Join *AB*, and at *A* and *B* apply two equal and opposite forces, each equal to *S*, and acting in the directions *BA* and *AB* respectively. Let these forces be represented by *AD* and *BE* respectively; they balance one another and have no effect on the equilibrium of the body.

Complete the parallelograms *ALFD* and *BMGE*, and produce the diagonals *AF* and *GB* to meet in *O*.

[These diagonals will always meet unless they be parallel, in which case the forces *P* and *Q* will be equal.]

Draw OC parallel to AL or BM to meet AB in C.

The forces P and S acting at A have a resultant P_1 represented by AF. Let its point of application be transferred to O.

So the forces Q and S acting at B have a resultant Q_1 represented by BG. Let its point of application be transferred to O.

The force P_1 at O may be resolved into two forces, S parallel to AD, and P in the direction CO produced.

So the forces Q_1 at O may be resolved into two forces, S parallel to BE, and Q in the direction OC.

Also these two forces S acting at O are in equilibrium.

Hence the original forces P and Q are equivalent to a force $P-Q$ acting in the direction CO produced, *i.e.* acting at C in a direction parallel to that of P.

To determine the position of the point C. The triangle OCA is, by construction, similar to the triangle FDA;

$$\therefore \frac{OC}{CA}=\frac{FD}{DA}=\frac{AL}{AD}=\frac{P}{S} \quad \text{............} (1).$$

So, since the triangles OCB, BMG are similar, we have

$$\frac{OC}{CB}=\frac{BM}{MG}=\frac{Q}{S} \quad \text{..................} (2).$$

Hence, from (1) and (2), by division, $\frac{CA}{CB}=\frac{Q}{P}$, *i.e.* C divides the line AB *externally* in the inverse ratio of the forces.

To sum up; If two parallel forces, P and Q, act at points A and B of a rigid body,

(i) their resultant is a force whose line of action is parallel to the lines of action of the component forces; also, when the component forces are like, its direction is the same as that of the two forces, and, when the forces are unlike, its direction is the same as that of the greater component.

(ii) the point of application is a point C in AB such that

$$P \cdot AC = Q \cdot BC.$$

(iii) the magnitude of the resultant is the sum of the two component forces when the forces are like, and the difference of the two component forces when they are unlike.

54. *Case of failure of the preceding construction.*

In the second figure of the last article, if the forces P and Q be equal, the triangles FDA and GEB are equal in all respects, and hence the angles DAF, EBG will be equal.

In this case the lines AF, GB will be parallel and will not meet in any such point as O; hence the construction fails.

Hence there is no single force which is equivalent to two equal unlike parallel forces.

We shall return to the consideration of this case in Chapter VI.

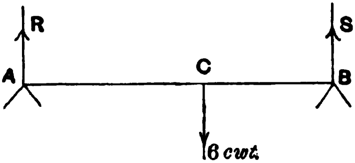
55. If we have a number of like parallel forces acting on a rigid body we can find their resultant by successive applications of Art. 53. We must find the resultant of the first and second, and then the resultant of this resultant and the third, and so on.

The magnitude of the final resultant is the sum of the forces.

If the parallel forces be not all like, the magnitude of the resultant will be found to be the algebraic sum of the forces each with its proper sign prefixed.

56. **Ex.** *A horizontal rod, 6 feet long, whose weight may be neglected, rests on two supports at its extremities; a body, of weight 6 cwt., is suspended from the rod at a distance of 2½ feet from one end; find the reaction at each point of support. If one support could only bear a pressure equal to the weight of 1 cwt., what is the greatest distance from the other support at which the body could be suspended?*

Let AB be the rod and R, S the pressures at the points of support. Let C be the point at which the body is suspended so that $AC=3\frac{1}{2}$ and $CB=2\frac{1}{2}$ feet. For equilibrium the resultant of R and S must balance 6 cwt. Hence, by Art. 53,



$$R+S=6 \qquad \text{(1)},$$

and

$$\frac{R}{S}=\frac{BC}{AC}=\frac{2\frac{1}{2}}{3\frac{1}{2}}=\frac{5}{7} \qquad \text{(2)}.$$

Solving (1) and (2), we have $R=\frac{5}{2}$, and $S=\frac{7}{2}$. Hence the pressures are 2½ and 3½ cwt. respectively.

If the reaction at A can only be equal to 1 cwt., S must be 5 cwt. Hence, if AC be x, we have

$$\frac{1}{5}=\frac{BC}{AC}=\frac{6-x}{x}.$$

$$\therefore\ x=5 \text{ feet}.$$

Hence BC is 1 foot.

EXAMPLES. VII.

In the 4 following examples A and B denote the points of application of parallel forces P and Q, and C is the point in which their resultant R meets AB.

✓ 1. Find the magnitude and position of the resultant (the forces being like) when

(i) $P=4$; $Q=7$; $AB=11$ inches;

(ii) $P=11$; $Q=19$; $AB=2\frac{1}{2}$ feet;

(iii) $P=5$; $Q=5$; $AB=3$ feet.

2. Find the magnitude and position of the resultant (the forces being unlike) when

(i) $P=17$; $Q=25$; $AB=8$ inches;
(ii) $P=23$; $Q=15$; $AB=40$ inches;
(iii) $P=26$; $Q=9$; $AB=3$ feet.

3. The forces being like,

(i) if $P=8$; $R=17$; $AC=4\frac{1}{2}$ inches; find Q and AB;
(ii) if $Q=11$; $AC=7$ inches; $AB=8\frac{3}{4}$ inches; find P and R;
(iii) if $P=6$; $AC=9$ inches; $CB=8$ inches; find Q and R.

4. The forces being unlike,

(i) if $P=8$; $R=17$; $AC=4\frac{1}{2}$ inches; find Q and AB;
(ii) if $Q=11$; $AC=-7$ inches; $AB=8\frac{3}{4}$ inches; find P and R;
(iii) if $P=6$; $AC=-9$ inches; $AB=12$ inches; find Q and R.

5. Find two like parallel forces acting at a distance of 2 feet apart, which are equivalent to a given force of 20 lbs. wt., the line of action of one being at a distance of 6 inches from the given force.

6. Find two unlike parallel forces acting at a distance of 18 inches apart which are equivalent to a force of 30 lbs. wt., the greater of the two forces being at a distance of 8 inches from the given force.

7. Two parallel forces, P and Q, act at given points of a body; if Q be changed to $\frac{P^2}{Q}$, shew that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

8. Two men carry a heavy cask of weight $1\frac{1}{2}$ cwt., which hangs from a light pole, of length 6 feet, each end of which rests on a shoulder of one of the men. The point from which the cask is hung is one foot nearer to one man than to the other. What is the pressure on each shoulder?

9. Two men, one stronger than the other, have to remove a block of stone weighing 270 lbs. by means of a light plank whose length is 6 feet; the stronger man is able to carry 180 lbs.; how must the block be placed so as to allow him that share of the weight?

10. A uniform rod, 12 feet long and weighing 17 lbs., can turn freely about a point in it and the rod is in equilibrium when a weight of 7 lbs. is hung at one end; how far from the end is the point about which it can turn?

N.B. *The weight of a uniform rod may be taken to act at its middle point.*

11. A straight uniform rod is 3 feet long; when a load of 5 lbs. is placed at one end it balances about a point 3 inches from that end; find the weight of the rod.

12. A uniform bar, of weight 3 lbs. and length 4 feet, passes over a prop and is supported in a horizontal position by a force equal to 1 lb. wt. acting vertically at the other end; find the distance of the prop from the centre of the beam.

13. A heavy uniform rod, 4 feet long, rests horizontally on two pegs which are one foot apart; a weight of 10 lbs. suspended from one end, or a weight of 4 lbs. suspended from the other end, will just tilt the rod up; find the weight of the rod and the distances of the pegs from the centre of the rod.

14. A uniform iron rod, $2\frac{1}{2}$ feet long and of weight 8 lbs., is placed on two rails fixed at two points, A and B, in a vertical wall. AB is horizontal and 5 inches long; find the distances at which the ends of the rod extend beyond the rails if the difference of the pressures on the rails be 6 lbs. wt.

15. A uniform beam, 4 feet long, is supported in a horizontal position by two props, which are 3 feet apart, so that the beam projects one foot beyond one of the props; shew that the pressure on one prop is double that on the other.

16. A straight weightless rod, 2 feet in length, rests in a horizontal position between two pegs placed at a distance of 3 inches apart, one peg being at one end of the rod, and a weight of 5 lbs. is suspended from the other end; find the pressure on the pegs.

17. One end of a heavy uniform rod, of weight W, rests on a smooth horizontal plane, and a string tied to the other end of the rod is fastened to a fixed point above the plane; find the tension of the string.

18. A man carries a bundle at the end of a stick which is placed over his shoulder; if the distance between his hand and his shoulder be changed how does the pressure on his shoulder change?

19. A man carries a weight of 50 lbs. at the end of a stick, 3 feet long, resting on his shoulder. He regulates the stick so that the length between his shoulder and his hands is (1) 12, (2) 18, (3) 24 inches; how great are the forces exerted by his hand and the pressures on his shoulder in each case?

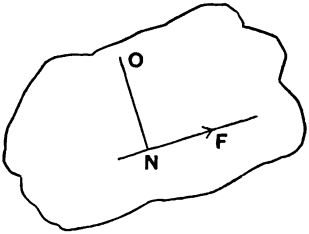
20. Three parallel forces act on a horizontal bar. Each is equal to 1 lb. wt., the right-hand one acting vertically upward and the other two vertically down at distances of 2 ft. and 3 ft. respectively from the first; find the magnitude and position of their resultant.

CHAPTER V.

MOMENTS.

57. **Def.** *The moment of a force about a given point is the product of the force and the perpendicular drawn from the given point upon the line of action of the force.*

Thus the moment of a force F about a given point O is

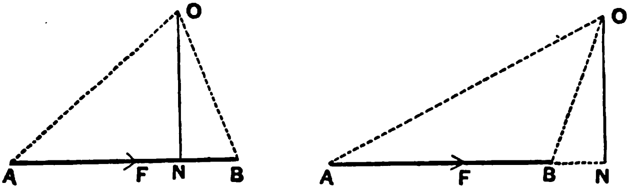


$F \times ON$, where ON is the perpendicular drawn from O upon the line of action of F.

It will be noted that the moment of a force F about a given point O never vanishes unless either the force vanishes or the force passes through the point about which the moment is taken.

58. *Geometrical representation of a moment.*

Suppose the force F to be represented in magnitude, direction, and line of action by the line AB. Let O be any given point and ON the perpendicular from O upon AB or AB produced.



Join OA and OB.

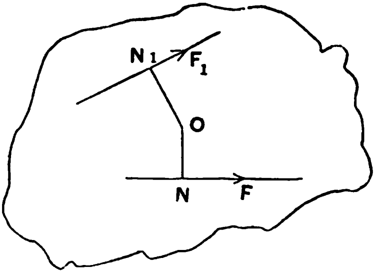
By definition the moment of F about O is $F \times ON$, *i.e.* $AB \times ON$. But $AB \times ON$ is equal to twice the area of the triangle OAB [for it is equal to the area of a rectangle whose base is AB and whose height is ON]. Hence the moment of the force F about the point O is represented by twice the area of the triangle OAB, *i.e. by twice the area of the triangle whose base is the line representing the force and whose vertex is the point about which the moment is taken.*

59. *Physical meaning of the moment of a force about a point.*

Suppose the body in the figure of Art. 57 to be a plane lamina [*i.e.* a body of very small thickness, such as a piece of sheet-tin or a thin piece of board] resting on a smooth table and suppose the point O of the body to be fixed. The effect of a force F acting on the body would be to cause it to turn about the point O as a centre, and this effect would not be zero unless (1) the force F were zero, or (2) the force F passed through O, in which case the distance ON would vanish. Hence the product $F \times ON$ would seem to be a fitting measure of the tendency of F to turn the body about O. This may be experimentally verified as follows;

Let the lamina be at rest under the action of two forces F and F_1, whose lines of action lie in the plane of the lamina. Let ON, ON_1 be the perpendiculars drawn from the fixed point O upon the lines of action of F and F_1.

If we measure the lengths ON, ON_1 and also the forces F and F_1, it will be found that the product $F . ON$ is always equal to the product $F_1 . ON_1$.



Hence the two forces, F and F_1, will have equal but opposite tendencies to turn the body about O if their moments about O have the same magnitude.

60. *Positive and negative moments.* In Art. 57 the force F would, if it were the only force acting on the lamina, make it turn in a direction opposite to that in which the hands of a watch move, when the watch is laid on the table with its face upwards.

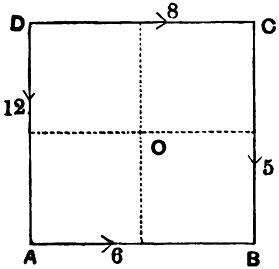
The force F_1 would, if it were the only force acting on the lamina, make it turn in the same direction as that in which the hands of the watch move.

The moment of F about O is said to be *positive*, and the moment of F_1 about O is said to be *negative*.

61. *Algebraic sum of moments.* The algebraic sum of the moments of a set of forces about a given point is the

sum of the moments of the forces, each moment having its proper sign prefixed to it.

Ex. ABCD *is a square; along the sides* AB, CB, DC, *and* DA *forces act equal respectively to* 6, 5, 8, *and* 12 *lbs. wt. Find the algebraic sum of their moments about the centre,* O, *of the square, if the side of the square be* 4 *feet.*



The forces along *DA* and *AB* tend to turn the square about *O* in the positive direction whilst the forces along the sides *DC*, *CB* tend to turn it in the negative direction.

The perpendicular distance of *O* from each force is 2 feet.

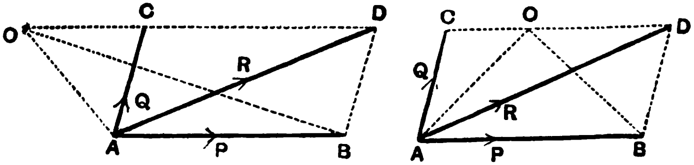
Hence the moments of the forces are respectively

$$+6 \times 2, \quad -5 \times 2, \quad -8 \times 2, \text{ and } +12 \times 2.$$

Their algebraic sum is therefore $2[6 - 5 - 8 + 12]$ or 10 units of moment, *i.e.* 10 times the moment of a force equal to 1 lb. wt. acting at the distance of 1 foot from *O*.

62. **Theorem.** *The algebraic sum of the moments of any two forces about any point in their plane is equal to the moment of their resultant about the same point.*

Case I. Let the forces meet in a point.



Let P, Q acting at the point A be the two forces and O the point about which the moments are taken. Draw OC parallel to the direction of P to meet the line of action of Q in the point C.

Let AC represent Q in magnitude and on the same scale let AB represent P; complete the parallelogram $ABDC$, and join OA and OB. Then AD represents the resultant, R, of P and Q.

(α) If O be without the angle DAC as in the first figure we have to shew that

$$2\triangle OAB + 2\triangle OAC = 2\triangle OAD.$$

Since AB and OD are parallel we have

$$\triangle OAB = \triangle DAB = \triangle ACD. \qquad \text{[Euc. I. 37]}$$

$$\therefore\ 2\triangle OAB + 2\triangle OAC = 2\triangle ACD + 2\triangle OAC = 2\triangle OAD.$$

(β) If O be within the angle CAD as in the second figure we have to shew that

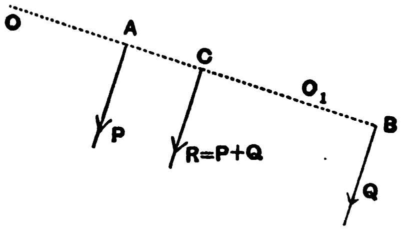
$$2\triangle AOB - 2\triangle AOC = 2\triangle AOD.$$

As in (α), we have

$$\triangle AOB = \triangle DAB = \triangle ACD.$$

$$\therefore\ 2\triangle AOB - 2\triangle AOC = 2\triangle ACD - 2\triangle OAC = 2\triangle OAD.$$

Case II. Let the forces be parallel.



Let P, Q be two parallel forces and $R\,(=P+Q)$ their resultant.

From any point O in their plane draw $OACB$ perpendicular to the forces to meet them in A, C, and B respectively.

By Art. 53 we have $P\,.\,AC=Q\,.\,CB$ ....(1)

$\therefore$ the sum of the moments of P and Q about O

$$\begin{aligned}&=Q\,.\,OB+P\,.\,OA\\&=Q\,(OC+CB)+P\,(OC-AC)\\&=(P+Q)\,OC+Q\,.\,CB-P\,.\,AC\\&=(P+Q)\,.\,OC,\text{ by equation (1)}\\&=\text{moment of the resultant about }O.\end{aligned}$$

If the point about which the moments are taken be between the forces, as O_1, the moments of P and Q have opposite signs.

In this case we have

Algebraic sum of moments of P and Q about O_1

$$\begin{aligned}&=P\,.\,O_1A-Q\,.\,O_1B\\&=P\,(O_1C+CA)-Q\,(CB-O_1C)\\&=(P+Q)\,.\,O_1C+P\,.\,CA-Q\,.\,CB\\&=(P+Q)\,.\,O_1C,\text{ by equation (1).}\end{aligned}$$

The case when the point has any other position, as also the case when the forces have opposite parallel directions, are left for the student to prove for himself.

63. Case I. of the preceding proposition may be otherwise proved in the following manner:

Let the two forces, P and Q, be represented by AB and AC respectively and let AD represent the resultant R so that $ABDC$ is a parallelogram.

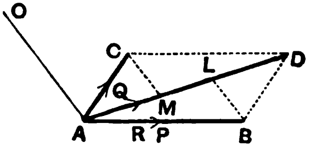
Let O be any point in the plane of the forces. Join OA and draw BL and CM, parallel to OA, to meet AD in L and M respectively.

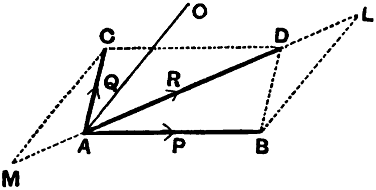
Since the sides of the triangle ACM are respectively parallel to the sides of the triangle DBL, and since AC is equal to BD,

$$\therefore\ AM=LD,$$

$$\therefore\ \triangle OAM=\triangle OLD. \qquad \text{[Euc. I. 38]}$$

First, let O fall without the angle CAD, as in the first figure.





Then

$$\begin{aligned}&2\triangle OAB+2\triangle OAC\\ &=2\triangle OAL+2\triangle OAM \qquad \text{[Euc. I. 37]}\\ &=2\triangle OAL+2\triangle OLD\\ &=2\triangle OAD.\end{aligned}$$

Hence the sum of the moments of P and Q is equal to that of R.

Secondly, let O fall within the angle CAD, as in the second figure.

The algebraic sum of the moments of P and Q about O

$$\begin{aligned}&=2\triangle OAB-2\triangle OAC\\ &=2\triangle OAL-2\triangle OAM \qquad \text{[Euc. I. 37]}\\ &=2\triangle OAL-2\triangle OLD\\ &=2\triangle OAD\\ &=\text{moment of } R \text{ about } O.\end{aligned}$$

64. If the point O about which the moments are taken lie on the resultant the moment of the resultant about the point vanishes. In this case the algebraic sum of the moments of the component forces about the given point

vanishes, i.e. *The moments of two forces about any point on the line of action of their resultant are equal and of opposite sign.*

The student will easily be able to prove this theorem independently from a figure; for, in Art. 62, the point O will be found to coincide with the point D and we have only to shew that the triangles ACO, ABO are now equal, and this is obviously true.

65. **Generalised theorem of moments.** *If any number of forces in one plane acting on a rigid body have a resultant, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant.*

For let the forces be P, Q, R, S,... and let O be the point about which the moments are taken.

Let P_1 be the resultant of P and Q,
P_2 be the resultant of P_1 and R,
P_3 be the resultant of P_2 and S,
and so on till the final resultant is obtained.

Then the moment of P_1 about $O =$ sum of the moments of P and Q (Art. 62);

Also the moment of P_2 about $O =$ sum of the moments of P_1 and R

$=$ sum of the moments of P, Q, and R.

So the moment of P_3 about O

$=$ sum of the moments of P_2 and S

$=$ sum of the moments of P, Q, R, and S,

and so on until all the forces have been taken.

Hence the moment of the final resultant

$=$ algebraic sum of the moments of the component forces.

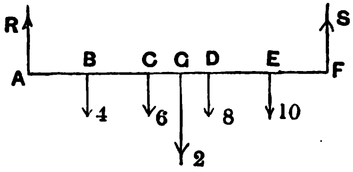
Cor. It follows, similarly as in Art. 64, that the algebraic sum of the moments of any number of forces about a point on the line of action of their resultant is zero; so, conversely, if the algebraic sum of the moments of any number of forces about any point in their plane vanishes, then, *either* their resultant is zero (in which case the forces are in equilibrium), *or* the resultant passes through the point about which the moments are taken.

66. The theorem of the previous article enables us to find points on the line of action of the resultant of a system of forces. For we have only to find a point about which the algebraic sum of the moments of the system of forces vanishes, and then the resultant must pass through that point. This principle is exemplified in Examples 2 and 3 of the following article.

If we have a system of parallel forces the resultant is known both in magnitude and direction when one such point is known.

67. Ex. 1. *A rod, 5 feet long, supported by two vertical strings attached to its ends has weights of* 4, 6, 8, *and* 10 *lbs. hung from the rod at distances of* 1, 2, 3, 4 *feet from one end. If the weight of the rod be* 2 *lbs., what are the tensions of the strings?*

Let AF be the rod, B, C, D, E the points at which the weights are



hung; let G be the middle point; we shall assume that the weight of the rod acts here.

Let R and S be the tensions of the strings. Since the resultant of the forces is zero, its moment about A must be zero.

Hence, by Art. 65, the algebraic sum of the moments about A must vanish.

Therefore $4 \times 1 + 6 \times 2 + 2 \times 2\frac{1}{2} + 8 \times 3 + 10 \times 4 - S \times 5 = 0,$

$$\therefore\ 5S = 4 + 12 + 5 + 24 + 40 = 85,$$

$$\therefore\ S = 17.$$

Similarly, taking moments about F, we have

$$5R = 10 \times 1 + 8 \times 2 + 2 \times 2\frac{1}{2} + 6 \times 3 + 4 \times 4 = 65,$$

$$\therefore\ R = 13.$$

The reaction R may be otherwise obtained. For the resultant of the weights is a weight equal to 30 lbs. and that of R and S is a force equal to $R+S$. But these resultants balance one another.

$$\therefore\ R + S = 30;$$

$$\therefore\ R = 30 - S = 30 - 17 = 13.$$

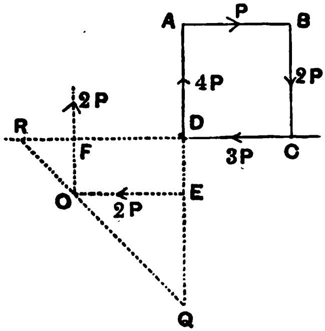
Ex. 2. *Forces equal to* P, 2P, 3P, *and* 4P *act along the sides of a square* ABCD *taken in order; find the magnitude, direction, and line of action of the resultant.*

Let the side of the square be a.

The forces P and $3P$ are, by Art. 53, equal to a parallel force $2P$ acting at E where DE is $\frac{a}{2}$.

The forces $4P$ and $2P$ are, similarly, equal to a force $2P$ acting at a point F on CD where DF is a.

Let the lines of action of these two components meet in O. Then the final resultant is equal to $2P\sqrt{2}$ acting in a direction parallel to CA.



Otherwise thus; without making any geometric construction (which is often tedious) the line of action of the resultant force can be easily obtained by using the theorem of Art. 65.

Let the line of action meet AD, CD in Q and R.

Since Q is a point on the line of action of the resultant the algebraic sum of the moments of the four forces about Q must be zero;

$$\therefore\ P\,(DQ+a)+2P\,(a)=3P\,.\,DQ\,;$$

$$\therefore\ DQ=\frac{3a}{2}.$$

So for the point R we have

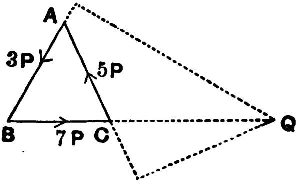
$$P\,.\,a+2P\,(RD+a)=4P\,.\,RD\,;$$

$$\therefore\ RD=\frac{3a}{2}.$$

Also the components of the forces perpendicular to CD are $4P-2P$, *i.e.* $2P$, and the components parallel to CD are $3P-P$, *i.e.* $2P$. Hence the magnitude of the resultant is $2\sqrt{2}P$.

Ex. 3. *Forces equal to* 3P, 7P, *and* 5P *act along the sides* AB, BC, CA *of an equilateral triangle* ABC; *find the magnitude, direction, and line of action of the resultant.*

Let the side of the triangle be a, and let the resultant force meet



the side BC in Q. Then, by Art. 65, the moments of the forces about Q vanish.

$$\therefore\ 3P\times(QC+a)\sin 60^\circ=5P\times QC\sin 60^\circ.$$

$$\therefore\ QC=\frac{3a}{2}.$$

The components of the forces perpendicular to BC

$$=5P\sin 60^\circ-3P\sin 60^\circ=P\sqrt{3}.$$

Also the components in the direction BC

$$=7P-5P\cos 60^\circ-3P\cos 60^\circ=3P.$$

Hence the resultant is $P\sqrt{12}$ inclined at an angle $\tan^{-1}\frac{\sqrt{3}}{3}$, *i.e.* 30°, to BC and passing through Q where $CQ=\frac{3}{2}BC$.

EXAMPLES. VIII.

1. The side of a square *ABCD* is 4 feet; along the lines *CB*, *BA*, *DA*, *DB*, respectively act forces equal to 4, 3, 2, and 5 lbs. weight; find to the nearest decimal of a foot-pound the algebraic sum of the moments of the forces about *C*.

2. The side of a regular hexagon *ABCDEF* is 2 feet; along the sides *AB*, *CB*, *DC*, *DE*, *EF*, *FA* act forces respectively equal to 1, 2, 3, 4, 5, 6 lbs. wt.; find the algebraic sum of the moments of the forces about *A*.

3. A pole of 20 feet length is placed with its end on a horizontal plane and is pulled by a string, attached to its upper end and inclined at 30° to the horizon, whose tension is equal to 30 lbs. wt.; find the horizontal force which applied at a point 4 feet above the ground will keep the pole in a vertical position.

4. A uniform iron rod is of length 6 feet and mass 9 lbs., and from its extremities are suspended masses of 6 and 12 lbs. respectively; from what point must the rod be suspended so that it may remain in a horizontal position?

5. A uniform beam is of length 12 feet and weight 50 lbs., and from its ends are suspended bodies of weights 20 and 30 lbs. respectively; at what point must the beam be supported so that it may remain in equilibrium?

6. Masses of 1 lb., 2 lbs., 3 lbs., and 4 lbs. are suspended from a uniform rod, of length 5 ft., at distances of 1 ft., 2 ft., 3 ft., and 4 ft. respectively from one end. If the mass of the rod be 4 lbs., find the position of the point about which it will balance.

7. A uniform rod, 4 ft. in length and weighing 2 lbs., turns freely about a point distant one foot from one end and from that end a weight of 10 lbs. is suspended. What weight must be placed at the other end to produce equilibrium?

8. A heavy uniform beam, 10 feet long, whose mass is 10 lbs., is supported at a point 4 feet from one end; at this end a mass of 6 lbs. is placed; find the mass which, placed at the other end, would give equilibrium.

9. The horizontal roadway of a bridge is 30 feet long, weighs 6 tons, and rests on similar supports at its ends. What is the pressure borne by each support when a carriage, of weight 2 tons, is (1) half-way across, (2) two-thirds of the way across?

10. A light rod, *AB*, 20 inches long, rests on two pegs whose distance apart is 10 inches. How must it be placed so that the pressures on the pegs may be equal when weights of $2W$ and $3W$ respectively are suspended from *A* and *B*?

11. A light rod, of length 3 feet, has equal weights attached to it, one at 9 inches from one end and the other at 15 inches from the other end; if it be supported by two vertical strings attached to its ends and if the strings cannot support a tension greater than the weight of 1 cwt., what is the greatest magnitude of the equal weights?

12. A heavy uniform beam, whose mass is 40 lbs., is suspended in a horizontal position by two vertical strings each of which can sustain a tension of 35 lbs. weight. How far from the centre of the beam must a body, of mass 20 lbs., be placed so that one of the strings may just break?

13. A uniform bar, *AB*, 10 feet long and of mass 50 lbs., rests on the ground. If a mass of 100 lbs. be laid on it at a point, distant 8 feet from *B*, find what vertical force applied to the end *A* will just begin to lift that end.

14. A rod, 16 inches long, rests on two pegs, 9 inches apart, with its centre midway between them. The greatest masses that can be suspended in succession from the two ends without disturbing the equilibrium are 4 lbs. and 5 lbs. respectively. Find the weight of the rod and the position of the point at which its weight acts.

15. A straight rod, 2 feet long, is movable about a hinge at one end and is kept in a horizontal position by a thin vertical string attached to the rod at a distance of 8 inches from the hinge and fastened to a fixed point above the rod; if the string can just support a mass of 9 ozs. without breaking, find the greatest mass that can be suspended from the other end of the rod, neglecting the weight of the rod.

16. A tricycle, weighing 5 stone 4 lbs., has a small wheel symmetrically placed 3 feet behind two large wheels which are 3 feet apart; if the centre of gravity of the machine be 9 inches behind the front wheels and that of the rider, whose weight is 9 stone, be 3 inches behind the front wheels, find the pressures on the ground of the different wheels.

17. A front-steering tricycle, of weight 6 stone, has a small wheel symmetrically placed 3 ft. 6 ins. in front of the line joining the two large wheels which are 3 feet apart; if the centre of gravity of the machine be distant horizontally 1 foot in front of the hind wheels and that of the rider, whose weight is 11 stone, be 6 inches in front of the hind wheels, find how the weight is distributed on the different wheels.

18. A dog-cart, loaded with 4 cwt., exerts a pressure on the horse's back equal to 10 lbs. wt.; find the position of the centre of gravity of the load if the distance between the pad and the axle be 6 feet.

19. Forces of 3, 4, 5 and 6 lbs. wt. respectively act along the sides of a square *ABCD* taken in order; find the magnitude, direction, and line of action of their resultant.

20. *ABCD* is a square; along *AB*, *CB*, *AD*, *DC* equal forces, *P*, act; find their resultant.

21. *ABCD* is a square the length of whose side is one foot; along *AB*, *BC*, *DC*, and *AD* act forces proportional to 1, 2, 4, 3 respectively; shew that the resultant is parallel to a diagonal of the square and find where it cuts the sides of the square.

22. *ABCD* is a rectangle of which adjacent sides *AB* and *BC* are equal to 3 and 4 feet respectively; along *AB*, *BC*, *CD* forces of 30, 40, and 50 lbs. wt. act; find the resultant.

23. Three forces *P*, $2P$, and $3P$ act along the sides *AB*, *BC*, and *CA* of a given equilateral triangle *ABC*; find the magnitude and direction of their resultant, and find also the point in which its line of action meets the side *BC*.

24. *ABC* is an isosceles triangle whose angle *A* is 120° and forces of magnitude 1, 1, and $\sqrt{3}$ lbs. wt. act along *AB*, *AC*, and *BC*; shew that the resultant bisects *BC* and is parallel to one of the other sides of the triangle.

25. Forces proportional to *AB*, *BC*, and $2CA$ act along the sides of a triangle *ABC* taken in order; shew that the resultant is represented in magnitude and direction by *CA* and that its line of action meets *BC* at a point *X* where *CX* is equal to *BC*.

26. *ABC* is a triangle and *D*, *E*, *F* are the middle points of the sides; forces represented by *AD*, $\frac{2}{3}BE$, $\frac{1}{3}CF$ act on a particle at the point where *AD* and *BE* meet; shew that the resultant is represented in magnitude and direction by $\frac{1}{2}AC$ and that its line of action divides *BC* in the ratio 2 : 1.

27. Three forces act along the sides of a triangle; shew that, if the sum of two of the forces be equal in magnitude but opposite in sense to the third force, then the resultant of the three forces passes through the centre of the inscribed circle of the triangle.

28. The wire passing round a telegraph pole is horizontal and the two portions attached to the post are inclined at an angle of 60° to one another. The pole is supported by a wire attached to the middle point of the post and inclined at 60° to the horizon; shew that the tension of this wire is $4\sqrt{3}$ times that of the telegraph wire.

29. At what height from the base of a pillar must the end of a rope of given length be fixed so that a given force acting at the other end may have the greatest tendency to make the pillar overturn?

30. The magnitude of a force is known and also its moments about two given points *A* and *B*. Find, by a geometrical construction, its line of action.

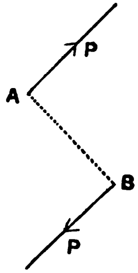
31. Find the locus of all points in a plane such that two forces given in magnitude and position shall have equal moments, in the same sense, round any one of these points.

32. *AB* is a diameter of a circle and *BP*, *BQ* are chords at right angles to one another; shew that the moments of forces represented by *BP* and *BQ* about *A* are equal.

CHAPTER VI.

COUPLES.

68. **Def.** Two equal unlike parallel forces form a couple.



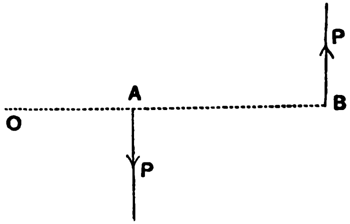
The Arm of a couple is the perpendicular distance between the lines of action of the two forces which form the couple, *i.e.* is the perpendicular drawn from any point lying on the line of action of one of the forces upon the line of action of the other. Thus the arm of the couple (P, P) is the length AB.

The Moment of a couple is the product of one of the forces forming the couple and the arm of the couple.

In the figure the moment of the couple is $P \times AB$.

69. **Theorem.** *The algebraic sum of the moments of the two forces forming a couple about any point in their plane is constant, and equal to the moment of the couple.*

Let the couple consist of two forces, each equal to P, and let O be any point in their plane.



Draw OAB perpendicular to the forces to meet them in A and B.

The algebraic sum of the moments of the forces about O

$$= P \, . \, OB - P \, . \, OA = P(OB - OA) = P \, . \, AB$$

= the moment of the couple, and is therefore the same whatever be the point O about which the moments are taken.

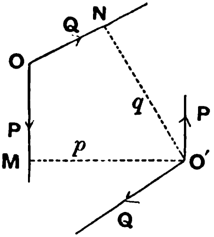
70. **Theorem.** *Two couples, acting in one plane upon a rigid body, whose moments are equal and opposite, balance one another.*

Let one couple consist of two forces (P, P), acting at the ends of an arm p, and let the other couple consist of two forces (Q, Q), acting at the ends of an arm q.

Case I. Let one of the forces P meet one of the forces Q in a point O, and let the other two forces meet in O'. From O' draw perpendiculars, $O'M$ and $O'N$, upon the forces which do not pass through O', so that these perpendiculars are p and q respectively.

Since the moments of the couples are equal in magnitude, we have

$$P \cdot p = Q \cdot q, \text{ i.e., } P \cdot O'M = Q \cdot O'N.$$



Hence, (Art. 64), O' is on the line of action of the resultant of P and Q acting at O, so that OO' is the direction of this resultant.

Similarly, the resultant of P and Q at O' is in the direction $O'O$.

Also these resultants are equal in magnitude; for the forces at O are respectively equal to, and act at the same angle as, the forces at O'.

Hence these two resultants destroy one another, and therefore the four forces composing the two couples are in equilibrium.

Case II. Let the forces composing the couples be all parallel, and let any straight line perpendicular to their directions meet them in the points A, B, C, D, as in the figure, so that we have

$$P \cdot AB = Q \cdot CD \text{.....................(i).}$$

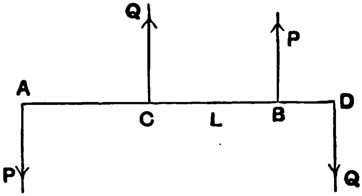
Let L be the point of application of the resultant of Q at C and P at B, so that

$$P \cdot BL = Q \cdot CL \text{....................(ii).}$$

By subtracting (ii) from (i), we have

$$P.AL = Q.LD,$$

so that L is the point of application of the resultant of P at A, and Q at D.



But the magnitude of each of these resultants is $(P+Q)$, and they have opposite directions; hence they are in equilibrium.

Therefore the four forces composing the two couples balance.

71. Since two couples in the same plane, of equal but opposite moment, balance, it follows, by reversing the directions of the forces composing one of the couples, that

Any two couples of equal moment in the same plane are equivalent.

It follows also that two like couples of equal moment are equivalent to a couple of double the moment.

72. **Theorem.** *Any number of couples in the same plane acting on a rigid body are equivalent to a single couple, whose moment is equal to the algebraic sum of the moments of the couples.*

For let the couples consist of forces (P, P) whose arm is p, (Q, Q) whose arm is q, (R, R) whose arm is r, etc. Replace the couple (Q, Q) by a couple whose components have the same lines of action as the forces (P, P). The magnitude of each of the forces of this latter couple will be $Q\frac{q}{p}$, since

$$Q\frac{q}{p} \cdot p = Q \cdot q.$$

So let the couple (R, R) be replaced by a couple $\left(R\frac{r}{p}, R\frac{r}{p}\right)$, whose forces act in the same lines as the forces (P, P).

Similarly for the other couples.

Hence all the couples are equivalent to a couple, each of whose forces is $P + Q\frac{q}{p} + R\frac{r}{p} + \ldots$ acting at an arm p.

The moment of this couple is

$$\left(P + Q\frac{q}{p} + R\frac{r}{p} + \ldots\right) \cdot p,$$

$$\text{i.e., } P \cdot p + Q \cdot q + R \cdot r + \ldots.$$

Hence the original couples are equivalent to a single couple, whose moment is equal to the sum of their moments.

If all the component couples have not the same sign we must give to each moment its proper sign, and the same proof will apply.

EXAMPLES. IX.

1. $ABCD$ is a square whose side is 2 feet; along AB, BC, CD, DA act forces equal to 1, 2, 8, and 5 lbs. wt., and along AC and DB forces equal to $5\sqrt{2}$ and $2\sqrt{2}$ lbs. wt.; shew that they are equivalent to a couple whose moment is equal to 16 foot-pound's weight.

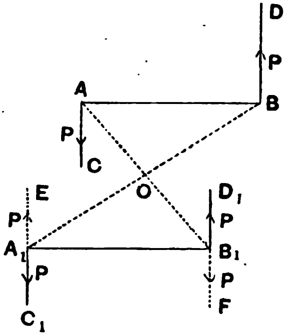
2. Along the sides AB, CD of a square $ABCD$ act forces each equal to 2 lbs. weight, whilst along the sides AD, CB act forces each equal to 5 lbs. weight; if the side of the square be 3 feet, find the moment of the couple that will give equilibrium.

3. *ABCDEF* is a regular hexagon; along the sides *AB*, *CB*, *DE*, *FE* act forces respectively equal to 5, 11, 5, and 11 lbs. weight, and along *CD* and *FA* act forces, each equal to x lbs. weight. Find x, if the forces be in equilibrium.

4. A horizontal bar *AB*, without weight, is acted upon by a vertical downward force of 1 lb. weight at *A*, a vertical upward force of 1 lb. weight at *B*, and a downward force of 5 lbs. weight at *C* inclined to the bar at an angle of 30°. Find at what point of the bar a force must be applied to balance these, and find also its magnitude and direction.

73. **Theorem.** *The effect of a couple upon a rigid body is unaltered if it be transferred to any plane parallel to its own, the arm remaining parallel to its original position.*

Let the couple consist of two forces (P, P), whose arm is AB, and let their lines of action be AC and BD.



Let A_1B_1 be any line equal and parallel to AB.

Draw A_1C_1 and B_1D_1 parallel to AC and BD respectively.

At A_1 introduce two equal and opposite forces, each equal to P, acting in the direction A_1C_1 and the opposite direction A_1E.

At B_1 introduce, similarly, two equal and opposite forces, each equal to P, acting in the direction B_1D_1 and the opposite direction B_1F.

These forces will have no effect on the equilibrium of the body.

Join AB_1 and A_1B, and let them meet in O; then O is the middle point of both AB_1 and A_1B.

The forces P at B and P acting along A_1E have a resultant $2P$ acting at O parallel to BD.

The forces P at A and P acting along B_1F have a resultant $2P$ acting at O parallel to AC.

These two resultants are equal and opposite, and therefore balance. Hence we have left the two forces (P, P) at A_1, B_1 acting in the directions A_1C_1 and B_1D_1, i.e., parallel to the directions of the forces of the original couple.

Also the plane through A_1C_1 and B_1D_1 is parallel to the plane through AC and BD.

Hence the theorem is proved.

Cor. From this proposition and Art. 71 we conclude that *A couple may be replaced by another couple acting in a parallel plane, provided that the moments of the two couples are the same.*

74. **Theorem.** *A single force and a couple acting in the same plane upon a rigid body cannot produce equilibrium, but are equivalent to the single force acting in a direction parallel to its original direction.*

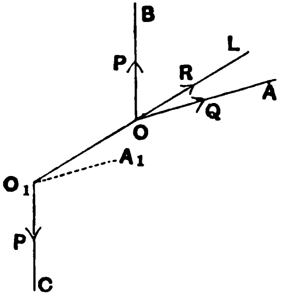
Let the couple consist of two forces, each equal to P, their lines of action being OB and O_1C respectively.

Let the single force be Q.

Case I. If Q be not parallel to the forces of the couple, let it be produced to meet one of them in O.

Then P and Q, acting at O, are equivalent to some force R, acting in some direction OL which lies between OA and OB.

Let OL be produced (backwards if necessary) to meet the other force of the couple in O_1, and let the point of application of R be transferred to O_1.



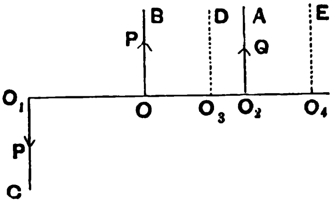
Draw O_1A_1 parallel to OA.

Then the force R may be resolved into two forces Q and P, the former acting in the direction O_1A_1, and the latter in the direction opposite to O_1C.

This latter force P is balanced by the second force P of the couple acting in the direction O_1C.

Hence we have left as the resultant of the system a force Q acting in the direction O_1A_1 parallel to its original direction OA.

Case II. Let the force Q be parallel to one of the forces of the couple.



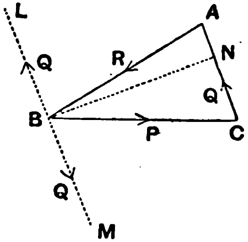
Let O_1O meet the force Q in O_2.

The parallel forces P at O and Q at O_2 are, by Art. 53, equivalent to a force $(P + Q)$ acting at some point O_3 in a direction parallel to OB. The unlike parallel forces $(P + Q)$ at O_3 and P at O_1 are, similarly, equivalent to a force Q acting at some point O_4 in a direction parallel to O_3D.

Hence the resultant of the system is equal to the single force Q acting in a direction parallel to its original direction.

75. *If three forces, acting upon a rigid body, be represented in magnitude, direction, and line of action by the sides of a triangle taken in order, they are equivalent to a couple whose moment is represented by twice the area of the triangle.*

Let ABC be the triangle and P, Q, R the forces, so that



P, Q, R are represented by the sides BC, CA, AB of the triangle.

Through B draw LBM parallel to the side AC, and introduce two equal and opposite forces, equal to Q, at B, acting in the directions BL and BM respectively. By the triangle of forces (Art. 36) the forces P, R, and Q acting in the straight line BL, are in equilibrium.

Hence we are left with the two forces, each equal to Q, acting in the directions CA and BM respectively.

These form a couple whose moment is $Q \times BN$, where BN is drawn perpendicular to CA.

Also $Q \times BN = CA \times BN =$ twice the area of the triangle ABC.

Cor. In a similar manner it may be shewn that if a system of forces acting on one plane on a rigid body be represented in magnitude, direction, and line of action by the sides of the polygon, they are equivalent to a couple whose moment is represented by twice the area of the polygon.

CHAPTER VII.

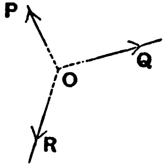
EQUILIBRIUM OF A RIGID BODY ACTED ON BY THREE FORCES IN A PLANE.

76. In the present chapter we shall discuss some simple cases of the equilibrium of a rigid body acted upon by three forces lying in a plane.

By the help of the theorem of the next article we shall find that the conditions of equilibrium reduce to those of a single particle.

77. **Theorem.** *If three forces, acting in one plane upon a rigid body, keep it in equilibrium, they must either meet in a point or be parallel.*

If the forces be not all parallel, at least two of them must meet; let these two be P and Q, and let their directions meet in O.



The third force R shall then pass through the point O.

Since the algebraic sum of the moments of any number of forces about a point in their plane is equal to the moment of their resultant,

∴ the sum of the moments of P, Q, and R about O is equal to the moment of their resultant.

But this resultant vanishes since the forces are in equilibrium.

Hence the sum of the moments of P, Q, and R about O is zero.

But, since P and Q both pass through O, their moments about O vanish.

Hence the moment of R about O vanishes.

∴ by Art. 57, since R is not zero, its line of action must pass through O.

Hence the forces meet in a point.

Otherwise. The resultant of P and Q must be some force passing through O.

But, since the forces P, Q, and R are in equilibrium, this resultant must balance R.

But two forces cannot balance unless they have the same line of action.

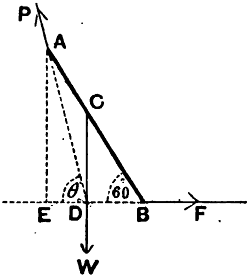
Hence the line of action of R must pass through O.

78. By the preceding theorem we see that the conditions of equilibrium of three forces, acting in one plane, are easily obtained. For the three forces must meet in a point; and by using Lami's Theorem, (Art. 40), or by resolving the forces in two directions at right angles, (Art. 48), we can obtain the required conditions.

79. **Ex. 1.** *A heavy uniform rod AB is hinged at A to a fixed point, and rests in a position inclined at 60° to the horizontal, being acted upon by a horizontal force F applied at the lower end B; find the action at the hinge and the magnitude of F.*

Let the vertical through C, the middle point of the rod, meet the horizontal line through B in the point D and let the weight of the rod be W.

There are only three forces acting on the rod, viz., the force F, the weight W, and the unknown reaction, P, of the hinge.



These three forces must therefore meet in a point.

Now F and W meet at D; hence the direction of the action at the hinge must be the line DA.

Draw AE perpendicular to EB, and let the angle ADE be θ.

Then $$\tan\theta=\frac{AE}{ED}=\frac{2AE}{EB}=2\tan 60^\circ=2\sqrt{3}.$$

Also, by Lami's Theorem,

$$\frac{F}{\sin WDA}=\frac{W}{\sin ADB}=\frac{P}{\sin WDB},$$

$$i.e.,\ \frac{F}{\sin(90^\circ+\theta)}=\frac{W}{\sin(180^\circ-\theta)}=\frac{P}{\sin 90^\circ}.$$

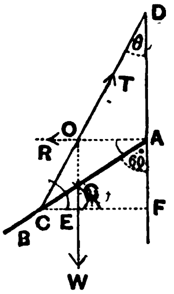
$$\therefore\ F=W\frac{\cos\theta}{\sin\theta}=W\cot\theta=\frac{W}{2\sqrt{3}}=\frac{W}{6}\sqrt{3},$$

and $$P=W\frac{1}{\sin\theta}=W\sqrt{1+\cot^2\theta}=W\sqrt{\frac{13}{12}}.$$

Ex. 2. *A uniform rod, AB, is inclined at an angle of 60° to the vertical with one end A resting against a smooth vertical wall, being supported by a string attached to a point C of the rod, distant 1 foot from B, and also to a ring in the wall vertically above A; if the length of the rod be 4 feet, find the position of the ring and the inclination and tension of the string.*

Let the perpendicular to the wall through A and the vertical line through the middle point, G, of the rod meet in O.

The third force, the tension T of the string, must therefore pass through O. Hence CO produced must pass through D, the position of the ring.



Let the angle CDA be θ, and draw CEF horizontal to meet OG in E and the wall in F.

Then
$$\tan\theta = \tan COE = \frac{CE}{OE} = \frac{CG\sin CGE}{AF}$$
$$= \frac{1 \,.\, \sin 60^\circ}{3 \,.\, \cos 60^\circ} = \frac{1}{\sqrt{3}}.$$
$$\therefore\ \theta = 30^\circ.$$
$$\therefore\ ACD = 60^\circ - \theta = 30^\circ.$$

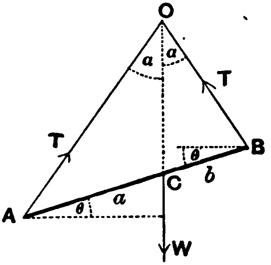
Hence $AD = AC = 3$ feet, giving the position of the ring.

If R be the reaction of the wall, and W be the weight of the beam, we have, since the forces are proportional to the sides of the triangle AOD,
$$\frac{T}{OD} = \frac{R}{AO} = \frac{W}{DA}.$$
$$\therefore\ T = W\frac{OD}{DA} = \frac{W}{\cos 30^\circ} = W \,.\, \frac{2}{\sqrt{3}},$$
and
$$R = W\frac{AO}{DA} = W\tan 30^\circ = W \,.\, \frac{1}{\sqrt{3}}.$$

Ex. 3. *A rod whose centre of gravity divides it into two portions, whose lengths are* a *and* b, *has a string, of length* l, *tied to its two ends and the string is slung over a small smooth peg; find the position of equilibrium of the rod.*

[N.B. *The centre of gravity of a body is the point at which its weight may be assumed to act.*]

Let AB be the rod and C its centre of gravity; let O be the peg and let the lengths of the portions AO, OB of the string be x and y respectively.



Since there are only three forces acting on the body they must meet in a point.

But the two tensions pass through O; hence the line of action of the weight W must pass through O, and hence the line CO must be vertical.

Now the tension T of the string is not altered, since the string passes round a smooth peg; hence, since W balances the resultant of two *equal* forces, it must bisect the angle between them.

$$\therefore \angle AOC = \angle BOC = \alpha \text{ (say)}.$$

Hence
$$\frac{x}{y} = \frac{AC}{CB} = \frac{a}{b} \qquad \text{(Euc. VI. 3).}$$

Also
$$x + y = l.$$

∴ solving these equations, we have

$$\frac{x}{a} = \frac{y}{b} = \frac{l}{a+b} \quad \text{.............................(i).}$$

Also, from the triangle AOB, we have

$$(a+b)^2 = x^2 + y^2 - 2xy\cos 2\alpha = (x+y)^2 - 2xy(1+\cos 2\alpha)$$
$$= (x+y)^2 - 4xy\cos^2\alpha = l^2 - 4\frac{l^2ab}{(a+b)^2}\cos^2\alpha.$$

$$\therefore \cos^2\alpha = \frac{l^2 - (a+b)^2}{4l^2}\,\frac{(a+b)^2}{ab} \quad \text{.................(ii).}$$

This equation gives α.

Let θ be the inclination of the rod to the horizon, so that

$$OCA = \frac{\pi}{2} + \theta.$$

From the triangle ACO we have

$$\frac{\sin\left(\frac{\pi}{2}+\theta\right)}{\sin\alpha}=\frac{AO}{AC}=\frac{x}{a}=\frac{l}{a+b}, \text{ by (i).}$$

$$\therefore \cos\theta=\frac{l\sin\alpha}{a+b}, \text{ giving } \theta.$$

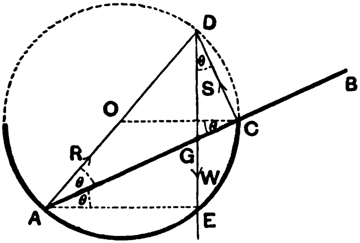
Also, by resolving the forces vertically, we have $2T\cos\alpha=W$, giving T.

Numerical Example. If the length of the rod be 5 feet, the length of the string 7 feet, and if the centre of gravity of the rod divide it in the ratio 4 : 3, shew that the portions of the string are at right angles, that the inclination of the rod to the horizon is $\tan^{-1}\frac{1}{7}$, and that the tension of the string is to the weight of the rod as $\sqrt{2}:2$.

Ex. 4. *A heavy uniform rod, of length* 2a, *rests partly within and partly without a fixed smooth hemispherical bowl, of radius* r; *the rim of the bowl is horizontal, and one point of the rod is in contact with the rim; if θ be the inclination of the rod to the horizon, shew that*

$$2r\cos 2\theta=a\cos\theta.$$

Let AB be the rod, G its centre of gravity, and C the point where the rod meets the edge of the bowl.



The reaction at A is along the line to the centre, O, of the bowl; for AO is the only line through A which is perpendicular to the surface of the bowl at A.

Also the reaction at C is perpendicular to the rod; for this is the only direction that is perpendicular to both the rod and the rim of the bowl.

These two reactions meet in a point D; also, by Euc. III. 31, D must lie on the geometrical sphere of which the bowl is a portion.

Hence the vertical line through G, the middle point of the rod, must pass through D.

Through A draw AE horizontal to meet DG in E and join OC.

Since OC and AE are parallel,

$$\therefore \angle OCA = \angle CAE = \theta.$$

Since $\quad OC = OA, \; \therefore \angle OAC = \angle OCA = \theta.$

Also $\quad \angle GDC = \frac{\pi}{2} - DGC = \theta.$

Now $\quad AE = AG \cos\theta = a\cos\theta,$

and $\quad AE = AD \cos 2\theta = 2r\cos 2\theta.$

$$\therefore \; 2r\cos 2\theta = a\cos\theta, \text{ giving } \theta.$$

Also, by Lami's Theorem, if R, S be the reactions at A and C, we have

$$\frac{R}{\sin\theta} = \frac{S}{\sin ADG} = \frac{W}{\sin ADC},$$

$$i.e., \; \frac{R}{\sin\theta} = \frac{S}{\cos 2\theta} = \frac{W}{\cos\theta}.$$

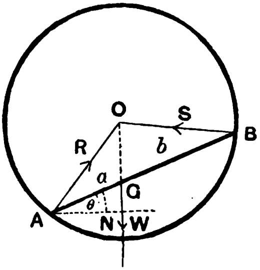
Numerical Example. If $r = \frac{\sqrt{3}}{2}a$, then we have $\theta = 30°$, and

$$R = S = W\frac{\sqrt{3}}{3}.$$

Ex. 5. *A beam whose centre of gravity divides it into two portions,* a *and* b, *is placed inside a smooth sphere; shew that, if* θ *be its inclination to the horizon in the position of equilibrium and* 2α *be the angle subtended by the beam at the centre of the sphere, then*

$$\tan\theta = \frac{b - a}{b + a}\tan\alpha.$$

In this case both the reactions, R and S, at the ends of the rod



pass through the centre, O, of the sphere. Hence the centre of gravity, G, of the rod must be vertically below O.

Let OG meet the horizontal line through A in N.

Now $$\angle OAB = \angle OBA = \frac{1}{2}(\pi - AOB) = \frac{\pi}{2} - \alpha.$$

$$\therefore \angle AOG = \frac{\pi}{2} - OAN = \frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha + \theta\right) = \alpha - \theta,$$

and $$\angle BOG = 2\alpha - AOG = \alpha + \theta.$$

Hence $$\frac{a}{b} = \frac{AG}{GB} = \frac{AG}{GO} \cdot \frac{GO}{GB}$$

$$= \frac{\sin AOG}{\sin OAG} \cdot \frac{\sin OBG}{\sin GOB} = \frac{\sin AOG}{\sin GOB}$$

$$= \frac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)} = \frac{\tan\alpha - \tan\theta}{\tan\alpha + \tan\theta}.$$

$$\therefore \frac{a+b}{b-a} = \frac{\tan\alpha}{\tan\theta}.$$

$$\therefore \tan\theta = \frac{b-a}{b+a}\tan\alpha.$$

This equation gives θ.

Also, by Lami's Theorem,

$$\frac{R}{\sin BOG} = \frac{S}{\sin AOG} = \frac{W}{\sin AOB}.$$

$$\therefore \frac{R}{\sin(\alpha + \theta)} = \frac{S}{\sin(\alpha - \theta)} = \frac{W}{\sin 2\alpha},$$

giving the reactions.

Numerical Example. If the rod be of weight 40 lbs., and subtend a right angle at the centre of the sphere, and if its centre of gravity divide it in the ratio 1 : 2, shew that its inclination to the horizon is $\tan^{-1}\frac{1}{3}$, and that the reactions are $8\sqrt{5}$ and $16\sqrt{5}$ lbs. weight respectively.

EXAMPLES. X.

1. A uniform rod, AB, of weight W, is movable in a vertical plane about a hinge at A, and is sustained in equilibrium by a weight P attached to a string BCP passing over a smooth peg C, AC being vertical; if AC be equal to AB, shew that $P = W \cos ACB$.

2. A uniform rod can turn freely about one of its ends, and is pulled aside from the vertical by a horizontal force acting at the other end of the rod equal to half its weight; at what inclination to the vertical will the rod rest?

3. A rod AB, hinged at A, is supported in a horizontal position by a string BC, making an angle of 45° with the rod, and the rod has a mass of 10 lbs. suspended from B. Neglecting the weight of the rod, find the tension of the string and the action at the hinge.

4. A uniform heavy rod AB has the end A in contact with a smooth vertical wall, and one end of a string is fastened to the rod at a point C, such that $AC=\frac{1}{4}AB$, and the other end of the string is fastened to the wall; find the length of the string, if the rod rest in a position inclined at an angle to the vertical.

5. ACB is a uniform rod, of weight W, such that $AC=\frac{1}{3}AB$; it is supported (B being uppermost) with its end A against a smooth vertical wall AB by means of a string CD, DB being horizontal and CD inclined to the wall at an angle of 30°. Find the tension of the string and the pressure on the wall.

6. A uniform rod, AB, resting with one end A against a smooth vertical wall is supported by a string BC which is tied to a point C vertically above A and to the other end B of the rod. Draw a diagram shewing the lines of action of the forces which keep the rod in equilibrium, and shew that the tension of the string is greater than the weight of the rod.

7. A uniform beam AB, of given length, is supported with its extremity, A, in contact with a smooth wall by means of a string CD fastened to a known point C of the beam and to a point D of the wall; if the inclination of the beam to the wall be given, shew how to find by geometrical construction the length of the string CD and the height of D above A.

For the problem to be possible, shew that the given angle BAD must be acute or obtuse according as AC is less or greater than $\frac{1}{2}AB$.

8. A rod, of length a, hangs against a smooth vertical wall being supported by means of a string, of length l, tied to one end of the rod, the other end of the string being attached to a point in the wall; shew that the rod can rest inclined to the wall at an angle θ given by $\cos^2\theta=\frac{l^2-a^2}{3a^2}$. What are the limits of the ratio of $a:l$ that equilibrium may be possible?

9. Equal weights P, P are attached to two strings ACP, BCP passing over a smooth peg C. AB is a heavy beam, of weight W, whose centre of gravity is a feet from A and b feet from B; shew that AB is inclined to the horizon at an angle

$$\tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right].$$

10. A heavy uniform beam is hung from a fixed point by two strings attached to its extremities; if the lengths of the strings and beam be as 2 : 3 : 4, shew that the tensions of the strings and the weight of the beam are as 2 : 3 : $\sqrt{10}$.

11. A heavy uniform rod, 15 inches long, is suspended from a fixed point by strings fastened to its ends, their lengths being 9 and 12 inches; if θ be the angle at which the rod is inclined to the vertical, shew that

$$25 \sin \theta = 24.$$

12. A straight uniform rod, of weight 3 lbs., is suspended from a peg by two strings, attached at one end to the peg and at the other to the extremities of the rod; the angle between the strings is a right angle and one is twice as long as the other; find their tensions.

13. Two equal heavy spheres, of 1 inch radius, are in equilibrium within a smooth spherical cup of 3 inches radius. Shew that the pressure between the cup and one sphere is double that between the two spheres.

14. A sphere, of given weight W, rests between two smooth planes, one vertical and the other inclined at a given angle α to the vertical; find the pressures on the planes.

15. A solid sphere rests upon two parallel bars which are in the same horizontal plane, the distance between the bars being equal to the radius of the sphere; find the pressure on each bar.

16. A smooth sphere is supported in contact with a smooth vertical wall by a string fastened to a point on its surface, the other end being attached to a point in the wall; if the length of the string be equal to the radius of the sphere, find the inclination of the string to the vertical, the tension of the string, and the pressure on the wall.

17. A picture of given weight, hanging vertically against a smooth wall, is supported by a string passing over a smooth peg driven into the wall; the ends of a string are fastened to two points in the upper rim of the frame which are equidistant from the centre of the rim, and the angle at the peg is 60°; compare the tension in this case with what it will be when the string is shortened to two-thirds of its length.

18. A picture, of 40 lbs. wt., is hung, with its upper and lower edges horizontal, by a cord fastened to the two upper corners and passing over a nail, so that the parts of the cord at the two sides of the nail are inclined to one another at an angle of 60°. Find the tension of the cord in lbs. weight.

19. A picture hangs symmetrically by means of a string passing over a nail and attached to two rings in the picture; what is the tension of the string when the picture weighs 10 lbs., if the string be 4 feet long and the nail distant 1 ft. 6 inches from the horizontal line joining the rings?

20. A picture frame, rectangular in shape, rests against a smooth vertical wall, from two points in which it is suspended by parallel strings attached to two points in the upper edge of the back of the frame, the length of each string being equal to the height of the frame. Shew that, if the centre of gravity of the frame coincide with its centre of figure, the picture will hang against the wall at an angle $\tan^{-1}\frac{b}{3a}$ to the vertical, where a is the height and b the thickness of the picture.

21. It is required to hang a picture on a vertical wall so that it may rest at a given inclination, α, to the wall and be supported by a string attached to a point in the wall at a given height h above the lowest edge of the picture; determine, by a geometrical construction, the point on the back of the picture to which the cord is to be attached and find the length of the cord that will be required.

22. A rod rests wholly within a smooth hemispherical bowl, of radius r, its centre of gravity dividing the rod into two portions of lengths a and b. Shew that, if θ be the inclination of the rod to the horizon in the position of equilibrium, then $\sin\theta = \frac{a-b}{2\sqrt{r^2-ab}}$, and find the pressure between the rod and the bowl.

23. In a smooth hemispherical-cup is placed a heavy rod, equal in length to the radius of the cup, the centre of gravity of the bowl being one-third of its length from one end; shew that the angle made by the rod with the vertical is $\tan^{-1}(3\sqrt{3})$.

24. A uniform rod, 4 inches in length, is placed with one end inside a smooth hemispherical bowl, of which the axis is vertical and the radius $\sqrt{3}$ inches; shew that a quarter of the rod will project over the edge of the bowl.

CHAPTER VIII.

GENERAL CONDITIONS OF EQUILIBRIUM OF A RIGID BODY ACTED ON BY FORCES IN ONE PLANE.

80. **Theorem.** *Any system of forces, acting in one plane upon a rigid body, can be reduced to either a single force or a single couple.*

By the parallelogram of forces any two forces, whose directions meet, can be compounded into one force; also, by Art. 53, two parallel forces can be compounded into one force provided they are not equal and unlike.

Take any two forces of the system and find their resultant, R_1; next find the resultant, R_2, of R_1 and the third force of the system; next, determine the resultant of R_2 and the fourth force of the system; and so on until all the forces have been exhausted.

Finally, we must either arrive at a single force, or we shall have two equal parallel unlike forces forming a couple.

81. **Theorem.** *If a system of forces act in one plane upon a rigid body, and if the algebraic sum of their moments about each of three points in the plane (not lying in the same straight line) vanish separately, the system of forces is in equilibrium.*

For any such system of forces, by the last article, reduces to either a single force or a single couple.

In our case they cannot reduce to a single couple; for, if they did, the sum of their moments about any point in their plane would, by Art. 69, be equal to a constant which is *not* zero, and this is contrary to our hypothesis.

Hence the system of forces cannot reduce to a single couple.

The system must therefore either be in equilibrium or reduce to a single force F.

Let the three points about which the moments are taken be A, B, and C.

Since the algebraic sum of the moments of a system of forces is equal to that of their resultant, therefore the moment of F about the point A must be zero.

Hence F is either zero, or passes through A.

Similarly, since the moment of F about B vanishes, F must be either zero or must pass through B, *i.e.*, F is either zero or acts in the line AB.

Finally, since the moment about C vanishes, F must be either zero or pass through C.

But (since the points A, B, C are *not* in the same straight line) the force cannot act along AB and also pass through C.

Hence the only admissible case is that F should be zero, *i.e.*, that the forces should be in equilibrium.

82. **Theorem.** *A system of forces, acting in one plane upon a rigid body, is in equilibrium, if the sum of their components parallel to each of two lines in their plane be zero, and if the algebraic sum of their moments about any point be zero also.*

For any such system of forces, by Art. 80, can be reduced to either a single force or a single couple.

In our case they cannot reduce to a single force.

For, since the sums of the components of the forces parallel to two lines in their plane are separately zero, therefore the components of their resultant force parallel to these two lines are zero also, and therefore the resultant force vanishes.

Neither can the forces reduce to a single couple; for, if they did, the moment of this couple about any point in its plane would be equal to a constant which is *not* zero; this, however, is contrary to our hypothesis.

Hence the system of forces must be in equilibrium.

83. It will be noted that in the enunciation of the last article nothing is said about the directions in which we are to resolve. In practice, however, it is almost always desirable to resolve along two directions at right angles.

Hence the conditions of equilibrium of any system of forces, acting in one plane upon a rigid body, may be obtained as follows;

I. **Equate to zero the algebraic sum of the resolved parts of all the forces in some fixed direction.**

II. **Equate to zero the algebraic sum of the resolved parts of all the forces in a perpendicular direction.**

III. **Equate to zero the algebraic sum of the moments of the forces about any point in their plane.**

The above three statical relations, together with the geometrical relations holding between the component portions of a system, will, in general, be sufficient to determine the equilibrium of any system acted on by forces which are in one plane.

In applying the preceding conditions of equilibrium to any particular case, great simplifications can often be introduced into the equations by properly choosing the directions along which we resolve. In general, the horizontal and vertical directions are the most suitable.

Again, the position of the point about which we take moments is important; it should be chosen so that as few of the forces as possible are introduced into the equation of moments.

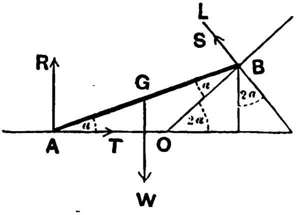
84. We have shewn that the conditions given in the previous article are sufficient for the equilibrium of the system of forces; they are also necessary.

Suppose we knew only that the first two conditions were satisfied. The system of forces might then reduce to a single couple; for the forces of this couple, being equal and opposite, are such that their components in any direction would vanish. In this case the forces would not be in equilibrium unless the third condition were satisfied.

Suppose, again, that we knew only that the components of the system along one given line vanished and that the moments about a given point vanished also; in this case the forces might reduce to a single force through the given point perpendicular to the given line; hence we see that it is necessary to have the sum of the components parallel to another line zero also.

85. **Ex. 1.** *A heavy uniform beam rests with one end upon a horizontal plane, and the other end upon a given inclined plane; it is kept in equilibrium by a string which is attached to the end resting on the horizontal plane and to the intersection of the inclined and horizontal planes; given that the inclination* (α) *of the beam to the horizontal is one half that of the inclined plane, find the tension of the string and the reactions of the planes.*

Let AB be the beam, AO the horizontal, and OB the inclined plane.



Let T be the tension of the string AO, W the weight of the body, R and S the reactions at A and B respectively vertical and perpendicular to OB.

Resolving horizontally and vertically we have

$$T = S\sin 2\alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$W = R + S\cos 2\alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, taking moments about A, we have

$$W\,.\,a\cos\alpha = S\,.\,AB\sin ABL = S\,.\,2a\cos\alpha \quad\ldots\ldots\ldots (3),$$

where $2a$ is the length of the beam.

These three equations give the circumstances of the equilibrium.

From (3), we have $S = \frac{1}{2}W$.

$\therefore$ from (2), $R = W - \frac{1}{2}W\cos 2\alpha = W(1 - \frac{1}{2}\cos 2\alpha)$.

Also, from (1),

$$T = \frac{W}{2}\sin 2\alpha.$$

Hence the reactions and the tension of the string are determined.

Suppose that, instead of the inclination of the beam to the horizon being given, the length of the string were given ($=l$ say).

Let us assume the inclination of the beam to the horizon to be θ.

The equations (1) and (2) remain the same as before.

The equation of moments would be, however,

$$W\,.\,a\cos\theta = S\,.\,AB\sin ABL = S\,.\,2a\cos ABO$$
$$= S\,.\,2a\cos(2\alpha - \theta) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

We should have a geometrical equation to determine θ, viz.,

$$\frac{l}{2a} = \frac{OA}{AB} = \frac{\sin ABO}{\sin AOB} = \frac{\sin(2\alpha - \theta)}{\sin 2\alpha} \quad\ldots\ldots\ldots\ldots (5).$$

This latter equation determines θ, and then the equations (1), (2), and (4) would give T, R, and S.

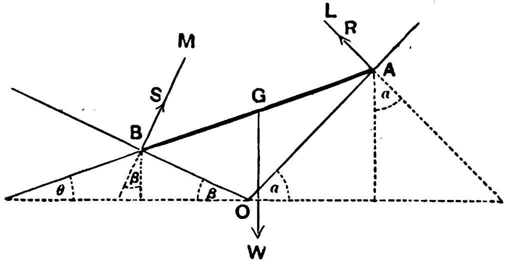
This question might have been solved by resolving along and perpendicular to the beam; in each equation we should then have involved each of the quantities T, R, S, and W, so that the resulting equations would have been more complicated than those above.

It was also desirable to take moments about A; for this is the only convenient point in the figure through which pass two of the forces which act on the body.

Ex. 2. *A beam whose centre of gravity divides it into portions, of lengths* a *and* b *respectively, rests in equilibrium with its ends resting on two smooth planes inclined at angles* α *and* β *respectively to the horizon, the planes intersecting in a horizontal line; find the inclination of the beam to the horizon and the pressures on the planes.*

Let the planes be OA and OB, and let AB be the rod, whose centre of gravity is G, so that GA and GB are a and b respectively.

Let R, S be the pressures at A and B perpendicular to the inclined planes, and let θ be the inclination of the beam to the horizon.



Resolving vertically and horizontally, we have

$$R\cos\alpha + S\cos\beta = W \qquad (1),$$

$$R\sin\alpha = S\sin\beta \qquad (2).$$

Also, by taking moments about G, we have

$$R\,.\,GA\sin GAL = S\,.\,GB\sin GBM.$$

Now
$$\angle GAL = \frac{\pi}{2} - BAO = \frac{\pi}{2} - (\alpha - \theta),$$

and
$$\angle GBM = \frac{\pi}{2} - ABO = \frac{\pi}{2} - (\beta + \theta).$$

Hence the equation of moments becomes

$$R\,.\,a\cos(\alpha - \theta) = S\,.\,b\cos(\beta + \theta) \qquad (3).$$

From (2), we have

$$\frac{R}{\sin\beta} = \frac{S}{\sin\alpha} = \frac{R\cos\alpha + S\cos\beta}{\sin\beta\cos\alpha + \sin\alpha\cos\beta} = \frac{W}{\sin(\alpha+\beta)}, \text{ by (1).}$$

These equations give R and S; also substituting for R, S in (3) we have

$$a\sin\beta\cos(\alpha - \theta) = b\sin\alpha\cos(\beta + \theta);$$

$$\therefore\ a\sin\beta(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = b\sin\alpha(\cos\beta\cos\theta - \sin\beta\sin\theta);$$

$$\therefore\ \sin\alpha\sin\beta\,(a+b)\sin\theta = \cos\theta\,(b\sin\alpha\cos\beta - a\cos\alpha\sin\beta);$$

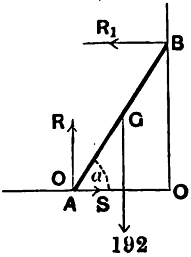
$$\therefore\ (a+b)\tan\theta = b\cot\beta - a\cot\alpha,$$

giving the value of θ.

Since there are only three forces acting on the body this question might have been solved by the methods of the last chapter.

Ex. 3. *A ladder, whose weight is 192 lbs. and whose length is 25 feet, rests with one end against a smooth vertical wall and with the other end upon the ground; if it be prevented from slipping by a peg at its lowest point, and if the lowest point be distant 7 feet from the wall, find the pressures on the peg, the ground, and the wall.*

Let AB be the rod and G its middle point; let R, R_1 be the pressures of the ground and wall, and S the horizontal pressure of the peg. Let the angle GAO be a, so that



$$\cos a = \frac{AO}{AB} = \frac{7}{25}.$$

Hence $$\sin a = \sqrt{1 - \frac{49}{625}} = \sqrt{\frac{576}{625}} = \frac{24}{25}.$$

Equating to zero the horizontal and vertical components of the forces acting on the rod, we have

$$R - 192 = 0 \quad\text{.............................. (1)},$$

$$R_1 - S = 0 \quad\text{.............................. (2)}.$$

Also, taking moments about A, we have

$$192 \times AG \cos a = R_1 \times AB \sin a \quad\text{..................(3)};$$

$$\therefore\ R_1 = 192 \times \tfrac{1}{2} \cot a = 96 \times \tfrac{7}{24} = 28.$$

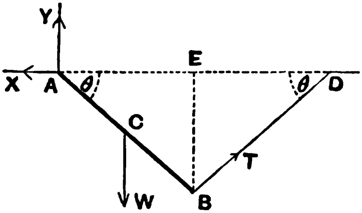
Hence, from (2), $S = 28$,

and, from (1), $R = 192$.

Hence the required pressures are 28, 192, and 28 lbs. weight respectively.

Ex. 4. *One end of a uniform rod is attached to a hinge, and the other end is supported by a string attached to the extremity of the rod, and the rod and string are inclined at the same angle, θ, to the horizontal; if W be the weight of the rod, shew that the action at the hinge is* $\frac{W}{4}\sqrt{8+\text{cosec}^2\theta}$.

Let AB be the rod, C its middle point, and BD the string meeting the horizontal line through A in D.



Let the tension of the string be T.

The action at the hinge is unknown both in magnitude and direction.

Let the horizontal and vertical components of this action be X and Y, as marked in the figure. Draw BE perpendicular to AD. Then $AD=2AE=2AB\cos\theta$.

Resolving horizontally and vertically, we have

$$X=T\cos\theta \quad\text{.......................(1)},$$

$$Y+T\sin\theta=W \quad\text{.......................(2)}.$$

Also, taking moments about A, we have

$$W.AC\cos\theta=T.AD\sin\theta=T.2AB\cos\theta\sin\theta \;\text{......}\;(3).$$

From (3), $$T=W\frac{AC}{2AB\sin\theta}=\frac{W}{4\sin\theta}.$$

Hence, by (1) and (2), we have

$$X=\frac{W}{4}\cot\theta,$$

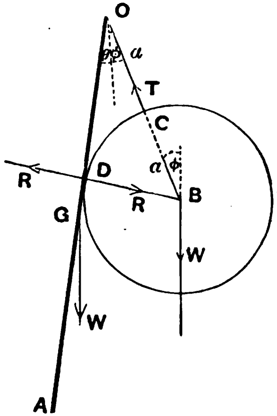
and $$Y=W-\frac{W}{4}=\frac{3W}{4}.$$

$\therefore$ the action at the hinge $=\sqrt{X^2+Y^2}$

$$=\frac{W}{4}\sqrt{9+\cot^2\theta}=\frac{W}{4}\sqrt{8+\text{cosec}^2\theta}.$$

Ex. 5. *A uniform heavy rod can turn freely about one end, which is fixed; to this end is attached a string which supports a sphere of radius* **a**. *If the length of the rod be* **4a**, *the length of the string* **a**, *and the weights of the sphere and rod be each W, find the inclinations of the rod and string to the vertical and the tension of the string.*

Let OA be the rod, OC the string, B the centre of the sphere, and D the point in which the rod touches the sphere.



Between the sphere and the rod at D there is a reaction, R, perpendicular to OD, acting in opposite directions on the two bodies.

Since there are only three forces acting on the sphere they must meet in a point, viz., the centre of the sphere. Hence OCB is a straight line.

Let θ, ϕ be the inclination of the rod and string to the vertical.

Then
$$\sin(\theta+\phi)=\frac{DB}{OB}=\frac{a}{2a}=\frac{1}{2},$$

so that
$$\theta+\phi=30^\circ \quad\text{.................................(1).}$$

The forces acting on the rod are the reaction at D, the weight of the rod, and the action at the hinge O.

If we take moments about O we shall avoid this action, and we have
$$W\,.\,2a\sin\theta=R\,.\,OD=R\,.\,2a\cos 30^\circ \quad\text{..............(2).}$$

From the conditions of equilibrium of the sphere, we have
$$\frac{T}{\sin(\phi+60^\circ)}=\frac{R}{\sin\phi}=\frac{W}{\sin 60^\circ} \quad\text{..................(3).}$$

∴ from (2) and (3), we have

$$\frac{\sin\theta}{\cos 30^\circ}=\frac{R}{W}=\frac{\sin\phi}{\sin 60^\circ}.$$

∴ $\phi=\theta$, and hence, from (1), we have

$$\theta=\phi=15^\circ.$$

Substituting in (3), we have

$$T=W\frac{\sin 75^\circ}{\sin 60^\circ}=W\frac{\sqrt{3}+1}{\sqrt{6}},$$

and

$$R=W\frac{\sin 15^\circ}{\sin 60^\circ}=W\frac{\sqrt{3}-1}{\sqrt{6}}.$$

EXAMPLES. XI.

1. A uniform beam, AB, whose weight is W, rests with one end, A, on a smooth horizontal plane AC. The other end, B, rests on a plane CB inclined to the former at an angle of 60°. If a string CA, equal to CB, prevent motion, find its tension.

2. A ladder, of weight W, rests with one end against a smooth vertical wall and with the other resting on a smooth floor; if the inclination of the ladder to the horizon be 60°, find the horizontal force that must be applied to the lower end to prevent the ladder from sliding down.

3. A beam, of weight W, is divided by its centre of gravity C into two portions AC and BC, whose lengths are a and b respectively. The beam rests in a vertical plane on a smooth floor AD and against a smooth vertical wall DB. A string is attached to a hook at D and to the beam at a point P. If T be the tension of the string, and θ, ϕ be the inclinations of the beam and string respectively to the horizon, shew that

$$T=W\frac{a\cos\theta}{(a+b)\sin(\theta+\phi)}.$$

4. A ladder rests at an angle α to the horizon, with its ends resting on a smooth floor and against a smooth vertical wall, the lower end being attached by a string to the junction of the wall and floor; find the tension of the string.

Find also the tension of the string when a man, whose weight is one-half that of the ladder, has ascended the ladder two-thirds of its length.

5. One end of a beam, of weight W, is placed on a smooth horizontal plane; the other end, to which a string is fastened, rests against another smooth plane inclined at an angle α to the horizon; the string, passing over a pulley at the top of the inclined plane, hangs vertically, and supports a weight P; shew that the beam will rest in all positions if $2P=W\sin\alpha$.

6. A heavy uniform beam rests with its extremities on two smooth inclined planes, which meet in a horizontal line, and whose inclinations to the horizon are α and β; find its inclination to the horizon in the position of equilibrium, and the pressures on the planes.

7. A uniform beam rests with a smooth end against the junction of the ground and a vertical wall, and is supported by a string fastened to the other end of the beam and to a staple in the wall. Find the tension of the string, and shew that it will be one-half the weight of the beam if the length of the string be equal to the height of the staple above the ground.

8. A uniform rod BC, of weight 2 lbs., can turn freely about B and is supported by a string, 8 inches long, attached to a point A in the same horizontal line as B, the distance AB being 10 inches. If the rod be 6 inches long, find the tension of AC.

9. A uniform rod has its upper end fixed to a hinge and its other end attached by a string to a fixed point in the same horizontal plane as the hinge, the length of the string being equal to the distance between the fixed point and the hinge. If the tension of the string be equal to the weight W of the rod, shew that the rod is inclined to the horizon at an angle $\tan^{-1}\frac{1}{2}$, and that the action of the hinge is equal to a force $\frac{W}{5}\sqrt{10}$ inclined at an angle $\tan^{-1}\frac{1}{3}$ to the horizon.

10. A rod is movable in a vertical plane about a hinge at one end, and at the other end is fastened a weight equal to half the weight of the rod; this end is fastened by a string, of length l, to a point at a height c vertically over the hinge. Shew that the tension of the string is $\frac{lW}{c}$, where W is the weight of the rod.

11. AB is a uniform rod, of length $8a$, which can turn freely about the end A, which is fixed; C is a smooth ring, whose weight is twice that of the rod, which can slide on the rod, and is attached by a string CD to a point D in the same horizontal plane as the point A; if AD and CD be each of length a, find the position of the ring and the tension of the string when the system is in equilibrium.

Shew also that the action on the rod at the fixed end A is a horizontal force equal to $\sqrt{3}W$, where W is the weight of the rod.

12. A rigid wire, without weight, in the form of the arc of a circle subtending an angle α at its centre, and having two weights P and Q at its extremities, rests with its convexity downwards upon a horizontal plane; shew that, if θ be the inclination to the vertical of the radius to the end at which P is suspended, then

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}.$$

13. A smooth hemispherical bowl, of radius a, is placed so that its edge touches a smooth vertical wall; a heavy rod is in equilibrium, inclined at 60° to the horizon, with one end resting on the inner surface of the bowl, and the other end resting against the wall; shew that the length of the rod must be $a + \frac{a}{\sqrt{13}}$.

14. A cylindrical vessel, of height 4 inches and diameter 3 inches, stands upon a horizontal plane, and a smooth uniform rod, 9 inches long, is placed within it resting against the edge. Find the pressure between the rod and the vessel, the weight of the former being 6 ounces.

15. A thin ring, of radius R and weight W, is placed round a vertical cylinder of radius r and prevented from falling by a nail projecting horizontally from the cylinder. Find the horizontal pressures between the cylinder and the ring.

16. A heavy carriage wheel, of weight W and radius r, is to be dragged over an obstacle, of height h, by a horizontal force F applied to the centre of the wheel; shew that

$$F = W\frac{\sqrt{2rh - h^2}}{r - h}.$$

17. A uniform beam, of length $2a$, rests in equilibrium, against a smooth vertical wall and upon a peg at a distance b from the wall; shew that the inclination of the beam to the vertical is

$$\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}.$$

18. A circular disc, BCD, of radius a and weight W, is supported by a smooth band, of inappreciable weight and thickness, which surrounds the disc along the arc BCD and is fastened at its extremities to the point A in a vertical wall, the portion AD touching the wall and the plane of the disc being at right angles to the wall. If the length of the band not in contact with the disc be $2b$, shew that the tension of the band is $\frac{W}{2}\frac{a^2+b^2}{b^2}$, and find the pressure at D.

19. Two equal uniform heavy straight rods are connected at one extremity by a string and rest upon two smooth pegs in the same horizontal line, one rod upon one peg and the other upon the other; if the distance between the pegs be equal to the length of each rod and the length of the string be half the same, shew that the rods rest at an angle θ to the horizon given by $2\cos^3\theta = 1$.

20. A uniform rod, whose weight is W, is supported by two fine strings, one attached to each end, which, after passing over small fixed smooth pulleys, carry weights w_1, w_2 respectively at the other ends. Shew that the rod is inclined to the horizon at an angle

$$\sin^{-1}\frac{w_1^2 - w_2^2}{W\sqrt{2(w_1^2 + w_2^2) - W^2}}.$$

21. A uniform rod, of weight W, is supported in equilibrium by a string, of length $2l$, attached to its ends and passing over a smooth peg. If a weight W' be now attached to one end of the rod, shew that a length $\frac{lW'}{W+W'}$ of the string will slip over the peg.

22. AB is a straight rod, of length $2a$ and weight λW, with the lower end A on the ground at the foot of a vertical wall AC, B and C being at the same vertical height $2b$ above A; a heavy ring, of weight W, is free to move along a string, of length $2l$, which joins B and C. If the system be in equilibrium with the ring at the middle point of the string, shew that

$$l^2=a^2-b^2\frac{\lambda(\lambda+2)}{(\lambda+1)^2}.$$

23. A given square board $ABCD$, of side b, is supported horizontally by two given loops of string $OACO$ and $OBDO$ passing under opposite corners and hung over a fixed hook O; find the tensions of the strings, if the height of O above the board be b.

24. A gate weighing 100 lbs. is hung on two hinges, 3 feet apart, in a vertical line which is distant 4 feet from the centre of gravity of the gate. Find the magnitude of the reactions at each hinge on the assumption that the whole of the weight of the gate is borne by the lower hinge.

25. A triangle, formed of three rods, is fixed in a horizontal position and a homogeneous sphere rests on it; shew that the pressure on each rod is proportional to its length.

26. A light triangular frame ABC stands in a vertical plane, C being uppermost, on two supports, A and B, in the same horizontal line and a mass of 18 lbs. weight is suspended from C. If $AB=AC=18$ feet, and $BC=5$ feet, find the pressures on the supports.

27. The sides of a triangular framework are 13, 20, and 21 inches in length; the longest side rests on a horizontal smooth table and a weight of 63 lbs. is suspended from the opposite angle. Find the tension in the side on the table.

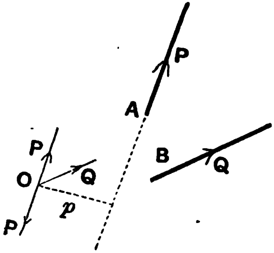
28. A bowl is formed from a hollow sphere, of radius a, and is so placed that the radius of the sphere drawn to each point in the rim makes an angle α with the vertical, whilst the radius drawn to a point A of the rim makes an angle β with the vertical; if a smooth uniform rod remain at rest with one end at A and a point of its length in contact with the rim, shew that the length of the rod is

$$4a\sin\beta\sec\frac{\alpha-\beta}{2}.$$

86. In the following articles the conditions of equilibrium enunciated in Art. 83 will be obtained in a slightly different manner.

*87. **Theorem.** *Any system of forces, acting in one plane upon a rigid body, is equivalent to a force acting at an arbitrary point of the body together with a couple.*

Let P be any force of the system acting at a point A of the body, and let O be any arbitrary point. At O introduce two equal and opposite forces, the magnitude of each being P, and let their line of action be parallel to that of P. These do not alter the state of equilibrium of the body.



The force P at A and the opposite parallel force P at O form a couple of moment $P.p$, where p is the perpendicular from O upon the line of action of the original force P.

Hence the force P at A is equivalent to a parallel force P at O and a couple of moment $P.p$.

So the force Q at B is equivalent to a parallel force Q at O and to a couple of moment $Q.q$, where q is the perpendicular from O on the line of action of Q.

The same holds for each of the system of forces.

Hence the original system of forces is equivalent to forces P, Q, R ... acting at O, parallel to their original directions, and a number of couples; these are equivalent to a single resultant force at O, and a single resultant couple of moment

$$P \,.\, p + Q \,.\, q + \ldots\ldots$$

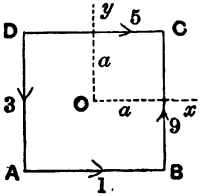
*88. By Art. 74 a force and a couple cannot balance unless each is zero.

Hence the resultant of P, Q, R, ... at O must be zero, and therefore, by Art. 48, *the sum of their resolved parts in two directions must separately vanish.*

Also the moment $Pp + Qq + \ldots$ must be zero, *i.e.*, *the algebraic sum of the moments of the forces about an arbitrary point O must vanish also.*

*89. **Ex.** *ABCD is a square; along the sides AB, BC, DC, DA act forces equal to* 1, 9, 5, *and* 3 *lbs. weight; find the force, passing through the centre of the square, and the couple which are together equivalent to the given system.*

Let O be the centre of the square and Ox, Oy perpendicular to the sides BC, CD respectively. Let the side of the square be $2a$.



The force 9 is equivalent to a force 9 along Oy together with a couple of moment $9 \,.\, a$.

The force 3 is equivalent to a force -3 along Oy together with a couple of moment $3 \,.\, a$.

The force 5 is equivalent to a force 5 along Ox together with a couple of moment $-5 \,.\, a$.

The force 1 is equivalent to a force 1 along Ox together with a couple of moment $1 \,.\, a$.

Hence the moment of the resultant couple is $9a + 3a - 5a + 1 \,.\, a$, i.e., $8 \,.\, a$.

The component force along Ox is 6 and the component along Oy is 6.

Hence the resultant force is one of $6\sqrt{2}$ lbs. weight inclined at 45° to the side AB.

EXAMPLES. XII.

1. A square is acted upon by forces equal to 2, 4, 6, and 8 lbs. weight along its sides taken in order; find the resultant force and the resultant couple of these forces, when the resultant force goes through the centre of the square.

2. $ABCD$ is a square along DA, AB, BC, CD, and DB act forces equal to P, $3P$, $5P$, $7P$, and $9\sqrt{2}P$; find the force, passing through A, and the couple, which are together equivalent to the system.

3. Forces equal to 1, 2, 3, 4, 5, and 6 lbs. weight respectively act along the sides AB, BC, CD, DE, EF, FA of a regular hexagon; find the force, passing through A, and the couple, which are together equivalent to the system.

4. Given in position a force equal to 10 lbs. weight and a couple consisting of two forces, each equal to 4 lbs. weight, at a distance of 2 inches asunder, draw the equivalent single force.

Constrained body.

90. A body is said to be constrained when one or more points of the body are fixed. For example, a rod attached to a wall by a ball-socket has one point fixed and is constrained.

If a rigid body have two points A and B fixed, all the points of the body in the line AB are fixed, and the only way in which the body can move is by turning round AB as an axis. For example, a door attached to the door-post by two hinges can only turn about the line joining the hinges.

If a body have three points in it fixed, it is plainly immovable.

The only cases we shall consider are (1) when the body has one point fixed and is acted upon by a system of forces lying in a plane passing through the fixed point, and (2) when the body can only move about a fixed axis in it and is acted upon by a system of forces whose directions are perpendicular to the axis.

91. *When a rigid body has one point fixed, and is acted upon by a system of forces in a plane passing through the point, it will be in equilibrium if the algebraic sum of the moments of the forces about the fixed point vanishes.*

When a body has one point O fixed, there must be exerted at the point some force of constraint, F, which together with the given system of forces is in equilibrium. Hence the conditions of equilibrium of Art. 83 must apply.

If we resolve along two directions at right angles, we shall have two equations to determine the magnitude and direction of the force F.

If we take moments about O for all the forces, the force F (since it passes through O) does not appear in our equation, and hence the equation of moments of Art. 83 will become an equation expressing the fact that the algebraic sum of the moments of the given system of forces about O is zero.

Hence for the equilibrium of the body (unless we wish to find the force of constraint F) we have only to express that the algebraic sum of the moments of the forces about the fixed point O is zero.

92. **Ex.** *A rod AB has one end A fixed, and is kept in a horizontal position by a force equal to* 10 *lbs. weight acting at B in a direction inclined at* 30° *to the rod; if the rod be homogeneous, and of length* 4 *feet, find its weight.*

The moment of the weight about A must be equal to the moment of the force about A.

If W be the weight, the former moment is $W \times 2$, and the latter is $10 \times 4 \sin 30°$.

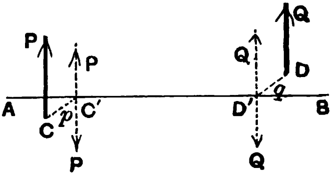
$$\therefore\ 2W = 10 \times 4 \sin 30° = 20.$$

$$\therefore\ W = 10 \text{ lbs. wt.}$$

93. *When a rigid body has an axis fixed, and is acted upon by forces, whose directions are perpendicular to this axis, it will be in equilibrium if the algebraic sum of the moments of the forces about the fixed axis vanishes.*

[If a force be perpendicular to a given axis and do not meet it, its moment about the axis is the product of the force and the perpendicular distance between the axis and the force.]

Suppose AB to be the fixed axis in the body, and let the body be acted on by forces P, $Q\ldots$, acting at points



C, D, ..., of the body, and let the directions of these forces be perpendicular to the axis.

Draw CC', DD', ... perpendicular to the axis, and let their lengths be p, q, ...

At C' introduce two equal and opposite forces, each equal to P, one of these being parallel to the original force P.

The force P at C and the two forces (P, P) at C' are equivalent to a force P, parallel to the original P, and a couple of moment $P \cdot p$.

Similarly, the force Q at D is equivalent to a force Q at D' and a couple of moment $Q \cdot q$.

Similarly for the other forces.

The forces, since they intersect the axis, can have no effect in turning the body about the axis and are balanced by the forces of constraint applied to the axis.

The couples are, by Art. 72, equivalent to a couple of moment $P \cdot p + Q \cdot q + \dots$ in a plane perpendicular to the axis.

Hence the body will be in equilibrium if

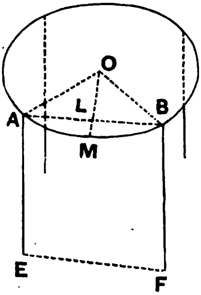
$$P \cdot p + Q \cdot q + \dots \text{ be zero;}$$

also the latter expression is the algebraic sum of the moments of the forces about the axis.

Hence the theorem is true.

94. **Ex.** *A circular uniform table, of weight* 80 *lbs., rests on four equal legs placed symmetrically round its edge; find the least weight which hung upon the edge of the table will just overturn it.*

Let AE, BF be two of the legs of the table, whose centre is O; the weight of the table will act through the point O.



If the weight be hung on the portion of the table between A and B the table will, if it turn at all, turn about the line joining the points E and F. Also it will be just on the point of turning when the weight and the weight of the table have equal moments about EF.

Now the weight will clearly have the greatest effect when placed at M, the middle point of the arc AB.

Let OM meet AB in L, and let x be the required weight. Taking moments about EF we have

$$x \cdot LM = 80 \cdot OL.$$

But
$$LM = OM - OL = OA - OA \cos 45^\circ$$
$$= OA\left(1 - \frac{1}{\sqrt{2}}\right).$$

$$\therefore \; x\left(1 - \frac{1}{\sqrt{2}}\right) OA = 80 \cdot OL = 80 \cdot \frac{1}{\sqrt{2}} \cdot OA,$$

and
$$x = \frac{80}{\sqrt{2} - 1} = 80(\sqrt{2} + 1)$$
$$= 193 \cdot 1 \text{ lbs. wt.}$$

95. **Theorem.** *If three forces acting on a body keep it in equilibrium, they must lie in a plane.*

Let the three forces be P, Q, and R.

Let P_1 and Q_1 be *any* two points on the lines of action of P and Q respectively.

Since the forces are in equilibrium, they can, taken together, have no effect to turn the body about the line P_1Q_1. But the forces P and Q meet this line, and therefore separately have no effect to turn the body about P_1Q_1. Hence the third force R can have no effect to turn the body about P_1Q_1.

Therefore the line P_1Q_1 must meet R.

Similarly, if Q_2, Q_3, ... be other points on the line of action of Q, the lines P_1Q_2, P_1Q_3, ... must meet R.

Hence R must lie in the plane through P_1 and the line of action of Q, *i.e.*, the lines of action of Q and R must be in a plane which passes through P_1.

But P_1 is *any* point on the line of action of P; and hence the above plane passes through *any* point on the line of action of P, *i.e.*, it contains the line of action of P.

EXAMPLES. XIII.

1. A square uniform plate is suspended at one of its vertices, and a weight, equal to half that of the plate, is suspended from the adjacent vertex of the square. Find the position of equilibrium of the plate.

2. A hollow vertical cylinder, of radius $2a$ and height $3a$, rests on a horizontal table, and a rod is placed within it with its lower end resting on the circumference of the base; if the weight of the rod be equal to that of the cylinder, how long must the rod be so that it may just cause the cylinder to topple over?

3. A cylinder, whose length is b and the diameter of whose base is c, is open at the top and rests on a horizontal plane; a uniform rod, of length $2a$, rests partly within the cylinder and in contact with it at its upper and lower edges; supposing the weight of the cylinder to be n times that of the rod, find the length of the rod when the cylinder is on the point of falling over.

4. A square table stands on four legs placed respectively at the middle points of its sides; find the greatest weight that can be put at one of the corners without upsetting the table.

5. A round table stands upon three equidistant weightless legs at its edge, and a man sits upon its edge opposite a leg. It just upsets and falls upon its edge and two legs. He then sits upon its highest point and just tips it up again. Shew that the radius of the table is $\sqrt{2}$ times the length of a leg.

6. A circular table, whose weight is 10 lbs., is provided with three vertical legs attached to three points in the circumference equidistant from one another; find the least weight which hung from any point in the edge of the table will just cause it to overturn.

7. A square four-legged table has lost one leg; where on the table should a weight, equal to the weight of the table, be placed, so that the pressures on the three remaining legs of the table may be equal?

8. A square table, of weight 20 lbs., has legs at the middle points of its sides, and three equal weights, each equal to the weight of the table, are placed at three of the angular points. What is the greatest weight that can be placed at the fourth corner so that equilibrium may be preserved?

9. A circular metallic plate, of uniform thickness and of weight w, is hung from a point on its circumference. A string wound on its edge, carries a weight p. Find the angle which the diameter through the point of suspension makes with the vertical.

10. A uniform circular disc, of weight nW, has a particle, of weight W, attached to a point on its rim. If the disc be suspended from a point A on its rim, B is the lowest point; also, if suspended from B, A is the lowest point. Shew that the angle subtended by AB at the centre of the disc is $2\sec^{-1} 2(n+1)$.

11. A heavy horizontal circular ring rests on three supports at the points A, B, and C of its circumference. Given its weight and the sides and angles of the triangle ABC, find the pressures on the supports.

CHAPTER IX.

CENTRE OF GRAVITY.

96. Every particle of matter is attracted to the centre of the Earth, and the force with which the Earth attracts any particle to itself is, as we shall see in Dynamics, proportional to the mass of the particle.

Any body may be considered as an agglomeration of particles.

If the body be small, compared with the Earth, the lines joining its component particles to the centre of the Earth will be very approximately parallel, and, within the limits of this book, we shall consider them to be absolutely parallel.

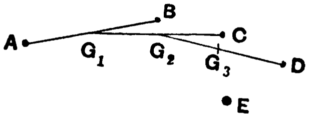
On every particle, therefore, of a rigid body there is acting a force vertically downwards which we call its weight.

These forces may by the process of compounding parallel forces, Art. 53, be compounded into a single force, equal to the sum of the weights of the particles, acting at some definite point of the body. Such a point is called the centre of gravity of the body.

Centre of gravity. Def. *The centre of gravity of a body, or system of particles rigidly connected together, is that point through which the line of action of the weight of the body always passes.*

97. *Every body, or system of particles rigidly connected together, has a centre of gravity.*

Let A, B, C, D... be a system of particles whose weights are w_1, w_2, w_3....



Join AB, and divide it at G_1 so that

$$AG_1 : G_1B :: w_2 : w_1.$$

Then parallel forces w_1 and w_2, acting at A and B, are, by Art. 53, equivalent to a force $(w_1 + w_2)$ acting at G_1.

Join G_1C, and divide it at G_2 so that

$$G_1G_2 : G_2C :: w_3 : w_1 + w_2.$$

Then parallel forces, $(w_1 + w_2)$ at G_1 and w_3 at C, are equivalent to a force $(w_1 + w_2 + w_3)$ at G_2.

Hence the forces w_1, w_2, w_3 may be supposed to be applied at G_2 without altering their effect.

Similarly, dividing G_2D in G_3 so that

$$G_2G_3 : G_3D :: w_4 : w_1 + w_2 + w_3,$$

we see that the resultant of the four weights at A, B, C, and D is equivalent to a vertical force, $w_1 + w_2 + w_3 + w_4$, acting at G_3.

Proceeding in this way, we see that the weights of any number of particles composing any body may be supposed to be applied at some point of the body without altering their effect.

98. Since the construction for the position of the resultant of parallel forces depends *only* on the point of application and magnitude, and *not* on the direction of the forces, the point we finally arrive at is the same if

the body be turned through any angle; for the weights of the portions of the body are still parallel, although they have not the same direction, relative to the body, in the two positions.

We can hence shew that a body can only have one centre of gravity. For, if possible, let it have two centres of gravity G and G_1. Let the body be turned, if necessary, until GG_1 be horizontal. We shall then have the resultant of a system of vertical forces acting both through G and through G_1. But the resultant force, being itself necessarily vertical, cannot act in the horizontal line GG_1.

Hence there can be only one centre of gravity.

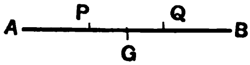
99. If the body be such that it is not so small that the weights of its component parts may all be considered to be very approximately parallel, it has not necessarily a centre of gravity.

In any case, the point of the body at which we arrive by the construction of Art. 97, has, however, very important properties and is called its Centre of Mass, or Centre of Inertia. It is also sometimes called the Centroid of the body.

100. We shall now proceed to the determination of the centre of gravity of some bodies of simple forms.

I. A uniform rod.

Let AB be a uniform rod, and G its middle point.



Take any point P of the rod between G and A, and a point Q in GB, such that

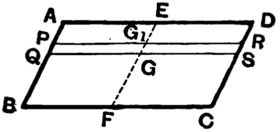
$$GQ = GP.$$

The centre of gravity of equal particles at P and Q is clearly G; also, for every particle between G and A, there is an equal particle at an equal distance from G, lying between G and B.

The centre of gravity of each of these pairs of particles is at G; therefore the centre of gravity of the whole rod is at G.

101. II. **A uniform parallelogram.**

Let $ABCD$ be a parallelogram, and let E and F be the middle points of AD and BC.



Divide the parallelogram into a very large number of strips, by means of lines parallel to AD, of which PR and QS are any consecutive pair. Then $PQSR$ may be considered to be a uniform straight line, whose centre of gravity is at its middle point G_1.

So the centre of gravity of all the other strips lies on EF, and hence the centre of gravity of the whole figure lies on EF.

So, by dividing the parallelogram by lines parallel to AB, we see that the centre of gravity lies on the line joining the middle points of the sides AB and CD.

Hence the centre of gravity is at G the point of intersection of these two lines.

G is clearly also the point of intersection of the diagonals of the parallelogram.

102. It is clear from the method of the two previous articles that, if in a uniform body we can find a point G such that the body can be divided into pairs of particles balancing about it, then G must be the centre of gravity of the body.

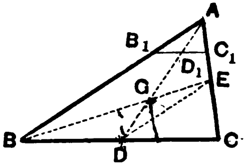
The centre of gravity of a uniform circle, or uniform sphere, is therefore its centre.

It is also clear that if we can divide a lamina into strips, the centres of gravity of which all lie on a straight line, then the centre of gravity of the lamina must lie on that line.

Similarly, if a body can be divided into portions, the centres of gravity of which lie in a plane, the centre of gravity of the whole must lie in that plane.

103. III. **Uniform triangular lamina.**

Let ABC be the triangular lamina and let D and E be the middle points of the sides BC and CA. Join AD and BE, and let them meet in G. Then G shall be the centre of gravity of the triangle.



Let B_1C_1 be *any* line parallel to the base BC meeting AD in D_1.

As in the case of the parallelogram, the triangle may be considered to be made up of a very large number of strips, such as B_1C_1, all parallel to the base BC.

Since B_1C_1 and BC are parallel, the triangles AB_1D_1 and ABD are similar; so also the triangles AD_1C_1 and ADC are similar.

Hence $$\frac{B_1D_1}{BD} = \frac{AD_1}{AD} = \frac{D_1C_1}{DC}.$$

But $BD = DC$; therefore $B_1D_1 = D_1C_1$. Hence the centre of gravity of the strip B_1C_1 lies on AD.

So the centres of gravity of all the other strips lie on AD, and hence the centre of gravity of the triangle lies on AD.

Join BE, and let it meet AD in G.

By dividing the triangle into strips parallel to AC we see, similarly, that the centre of gravity lies on BE.

Hence the required centre of gravity must be at G.

Since D is the middle point of BC and E is the middle point of CA, therefore DE is parallel to AB.

Hence the triangles GDE, GAB are similar,

$$\therefore \frac{GD}{GA} = \frac{DE}{AB} = \frac{CE}{CA} = \tfrac{1}{2},$$

so that $2GD = GA$, and $3GD = GA + GD = AD$.

$$\therefore GD = \tfrac{1}{3}AD.$$

Hence the centre of gravity of a triangle is on the line joining the middle point of any side to the opposite vertex at one-third the distance of the vertex from that side.

104. *The centre of gravity of any triangular lamina is the same as that of three equal particles placed at the vertices of the triangle.*

Taking the figure of Art. 103, the centre of gravity of two equal particles, each equal to w, at B and C, is at D the middle point of BC; also the centre of gravity of $2w$ at D and w at A divides the line DA in the ratio of $1 : 2$. But G, the centre of gravity of the lamina, divides DA in the ratio of $1 : 2$.

Hence the centre of gravity of the three particles is the same as that of the lamina.

105. IV. **Three rods forming a triangle.**

Let BC, CA, AB be the three rods forming the triangle and D, E, F the middle points of the rods. Join DE, EF, and FD. Clearly DE, EF, FD are half of AB, BC, CA respectively. The centres of gravity of the three rods are D, E, and F.

Proceeding in this manner we easily have

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + \ldots + w_n x_n}{w_1 + w_2 + \ldots + w_n},$$

whatever be the number of the particles in the system.

Otherwise, The above formula may be obtained by the use of Article 65. For the weights of the particles form a system of parallel forces whose resultant is equal to their sum, viz. $w_1 + w_2 + \ldots + w_n$. Also the sum of the moments of these forces about any point in their plane is the same as the moment of their resultant. But the sum of the moments of the forces about the fixed point O is

$$w_1 x_1 + w_2 x_2 + \ldots + w_n x_n.$$

Also, if $\bar{x}$ be the distance of the centre of gravity from O, the moment of the resultant is

$$(w_1 + w_2 + \ldots + w_n) \times \bar{x}.$$

Hence $$x\,(w_1 + w_2 + \ldots + w_n) = w_1 x_1 + w_2 x_2 + \ldots + w_n x_n;$$

i.e., $$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + \ldots + w_n x_n}{w_1 + w_2 + \ldots + w_n}.$$

110. **Ex. 1.** *A rod* AB, 2 *feet in length, and of weight* 5 *lbs., is trisected in the points* C, D, *and at the points* A, C, D, B *are placed particles of* 1, 2, 3, 4 *lbs. weight respectively; find what point of the rod must be supported so that the rod may rest in any position, i.e., find the centre of gravity of the system.*

Let G be the middle point of the rod and let the fixed point O of the previous article be taken to coincide with the end A of the rod. The quantities x_1, x_2, x_3, x_4, x_5 are in this case 0, 8, 12, 16, and 24 inches respectively.

Hence, if X be the point required, we have

$$AX = \frac{1 \cdot 0 + 2 \cdot 8 + 5 \cdot 12 + 3 \cdot 16 + 4 \cdot 24}{1 + 2 + 5 + 3 + 4}$$

$$= \frac{220}{15} = 14\tfrac{2}{3} \text{ inches.}$$

Ex. 2. *If, in the previous question, the body at* B *be removed and another body be substituted, find the weight of this unknown body so that the new centre of gravity may be at the middle point of the rod.*

Let λ lbs. be the required weight.

Since the distance of the new centre of gravity from A is to be 12 inches, we have

$$12=\frac{1.0+2.8+5.12+3.16+\lambda.24}{1+2+5+3+\lambda}=\frac{124+24\lambda}{11+\lambda}.$$

$$\therefore\ 132+12\lambda=124+24\lambda.$$

$$\therefore\ \lambda=\tfrac{2}{3}\text{ lb.}$$

Ex. 3. *To the end of a rod, whose length is* 2 *feet and whose weight is* 3 *lbs., is attached a sphere, of radius* 2 *inches and weight* 10 *lbs.; find the position of the centre of gravity of the compound body.*

Let OA be the rod, G_1 its middle point, G_2 the centre of the sphere, and G the required point.

Then $$OG=\frac{3\,.\,OG_1+10\,.\,OG_2}{3+10}.$$

But $$OG_1=12\text{ inches; } OG_2=26\text{ inches.}$$

$$\therefore\ OG=\frac{3\,.\,12+10\,.\,26}{3+10}=\frac{296}{13}=22\tfrac{10}{13}\text{ inches.}$$

EXAMPLES. XV.

1. A straight rod, 1 foot in length and weighing 1 ounce, has an ounce of lead fastened to it at one end, and another ounce fastened to it at a distance from the other end equal to one-third of its length; find the centre of gravity of the system.

2. A uniform bar, 3 feet in length and of mass 6 ounces, has 3 rings, each of mass 3 ounces, at distances 3, 15 and 21 inches from one end. About what point of the bar will the system balance?

3. A uniform rod AB is 4 feet long and weighs 3 lbs. One lb. is attached at A, 2 lbs. at a point distant 1 foot from A, 3 lbs. at 2 feet from A, 4 lbs. at 3 feet from A, and 5 lbs. at B. Find the distance from A of the centre of gravity of the system.

4. A telescope consists of 3 tubes, each 10 inches in length, one within the other, and of weights 8, 7, and 6 ounces. Find the position of the centre of gravity when the tubes are drawn out at full length.

5. Twelve heavy particles at equal intervals of one inch along a straight rod weigh 1, 2, 3, ... 12 grains respectively; find their centre of gravity, neglecting the weight of the rod.

6. Weights proportional to 1, 4, 9, and 16 are placed in a straight line so that the distances between them are equal; find the position of their centre of gravity.

7. A rod, of uniform thickness, has one-half of its length composed of one metal and the other half composed of a different metal, and the rod balances about a point distant one-third of its whole length from one end; compare the weight of equal quantities of the metal.

8. An inclined plane, with an angle of inclination of 60°, is 3 feet long; masses of 7, 5, 4 and 8 ounces are placed on the plane in order at distances of 1 foot, the latter being the highest; find the distance of their centre of gravity from the base of the inclined plane.

9. AB is a uniform rod, of length n inches and weight $(n+1)W$. To the rod masses of weight $W, 2W, 3W, \dots nW$ are attached at distances 1, 2, 3... n inches respectively from A. Find the distance from A of the centre of gravity of the rod and weights.

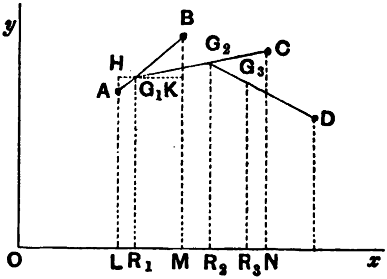
10. A rod, 12 feet long, has a mass of 1 lb. suspended from one end, and, when 15 lbs. is suspended from the other end, it balances about a point distant 3 ft. from that end; if 8 lbs. be suspended there, it balances about a point 4 ft. from that end. Find the weight of the rod and the position of its centre of gravity.

111. **Theorem.** *If a system of particles, whose weights are* $w_1, w_2, \dots w_n$, *lie in a plane, and if* Ox, Oy *be two fixed straight lines in the plane at right angles, and if the distances of the particles from* Ox *be* $y_1, y_2, \dots y_n$, *and the distance of their centre of gravity be* $\bar{y}$, *then*

$$\bar{y} = \frac{w_1y_1 + w_2y_2 + \dots + w_ny_n}{w_1 + w_2 + \dots + w_n}.$$

Similarly, if the distances of the particles from Oy *be* $x_1, x_2, \dots x_n$ *and that of their centre of gravity be* $\bar{x}$, *then*

$$x = \frac{w_1x_1 + w_2x_2 + \dots + w_nx_n}{w_1 + w_2 + \dots + w_n}.$$



Let A, B, C, ... be the particles, and AL, BM, CN... the perpendiculars on Ox.

Let G_1 be the centre of gravity of w_1 and w_2, G_2 the centre of gravity of $(w_1 + w_2)$ at G_1 and w_3 at C, and so on.

Draw G_1R_1, G_2R_2, ... perpendicular to Ox, and through G_1 draw HG_1K parallel to Ox to meet AL and BM in H and K.

Since G_1 is the centre of gravity of w_1 and w_2 we have

$$\frac{AG_1}{G_1B} = \frac{w_2}{w_1}, \quad \text{(Art. 97.)}$$

Now AG_1H and BG_1K are similar triangles,

$$\therefore \quad \frac{HA}{BK} = \frac{AG_1}{G_1B} = \frac{w_2}{w_1}.$$

But $\qquad HA = HL - AL = G_1R_1 - y_1;$

and $\qquad BK = BM - KM = y_2 - G_1R_1;$

$$\therefore \quad \frac{G_1R_1 - y_1}{y_2 - G_1R_1} = \frac{w_2}{w_1}.$$

Hence $\qquad w_1(G_1R_1 - y_1) = w_2(y_2 - G_1R_1);$

$$\therefore \quad G_1R_1 = \frac{w_1y_1 + w_2y_2}{w_1 + w_2} \quad \text{............ ...(1).}$$

Similarly, since G_2 is the centre of gravity of $(w_1 + w_2)$ at G_1 and w_3 at C, we have

$$G_2R_2 = \frac{(w_1 + w_2) \cdot G_1R_1 + w_3y_3}{w_1 + w_2 + w_3} = \frac{w_1y_1 + w_2y_2 + w_3y_3}{w_1 + w_2 + w_3}, \text{ by (1).}$$

Proceeding in this way we easily obtain

$$\bar{y} = \frac{w_1y_1 + w_2y_2 + \ldots + w_ny_n}{w_1 + w_2 + \ldots + w_n}.$$

Again, since the triangles AG_1H, BG_1K are similar, we have

$$\frac{HG_1}{G_1K} = \frac{AG_1}{G_1B} = \frac{w_2}{w_1}.$$

But
$$HG_1 = LR_1 = OR_1 - OL = OR_1 - x_1,$$
$$G_1K = R_1M = OM - OR_1 = x_2 - OR_1.$$
$$\therefore\ w_1(OR_1 - x_1) = w_2(x_2 - OR_1).$$

Hence
$$OR_1 = \frac{w_1x_1 + w_2x_2}{w_1 + w_2}.$$

Proceeding as before we finally have

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \dots + w_nx_n}{w_1 + w_2 + \dots + w_n}.$$

The theorem of this article may be put somewhat differently as follows;

The distance of the centre of gravity from **any** *line in the plane of the particles is equal to a fraction, whose numerator is the sum of the products of each weight into its distance from the given line, and whose denominator is the sum of the weights.*

112. The formula of the preceding article may be deduced from Article 93. For, since the resultant weight $(w_1 + w_2 + \dots + w_n)$ acting at G is equivalent to the component forces w_1, w_2, ... the resultant would, if the line Ox be supposed to be a fixed axis, have the same moment about this fixed axis that the component weights have.

But the moment of the resultant is

$$(w_1 + w_2 + \dots + w_n)\,\bar{y},$$

and the sum of the moments of the weights is

$$w_1y_1 + w_2y_2 + \dots + w_ny_n.$$

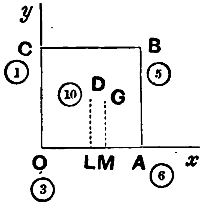
Hence
$$\bar{y} = \frac{w_1y_1 + w_2y_2 + \dots + w_ny_n}{w_1 + w_2 + \dots + w_n}.$$

In a similar manner we should have

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \dots + w_nx_n}{w_1 + w_2 + \dots + w_n}.$$

113. **Ex. 1.** *A square lamina, whose weight is* 10 *lbs., has attached to its angular points particles whose weights, taken in order, are* 3, 6, 5 *and* 1 *lbs. respectively. Find the position of the centre of gravity of the system, if the side of the lamina be* 25 *inches.*

Let the particles be placed at the angular points O, A, B, C. Let the two fixed lines from which the distances are measured be OA and OC.



The weight of the lamina acts at its centre D. Let G be the required centre of gravity and draw DL, GM perpendicular to Ox.

The distances of the points O, A, B, C, D from Ox are clearly 0, 0, 25, 25, and $12\frac{1}{2}$ inches respectively.

$$\therefore\ MG=\bar{y}=\frac{3.0+6.0+5.25+1.25+10.12\frac{1}{2}}{3+6+5+1+10}=\frac{275}{25}=11 \text{ ins.}$$

So the distances of the particles from Oy are 0, 25, 25, 0 and $12\frac{1}{2}$ inches respectively.

$$\therefore\ OM=\bar{x}=\frac{3.0+6.25+5.25+1.0+10.12\frac{1}{2}}{3+6+5+1+10}=\frac{400}{25}=16 \text{ ins.}$$

Hence the required point may be obtained by measuring 16 inches from O along OA and then erecting a perpendicular of length 11 inches.

Ex. 2. *OAB is an isosceles weightless triangle, whose base OA is 6 inches and whose sides are each 5 inches; at the points O, A, B are placed particles of weights 1, 2, and 3 lbs.; find their centre of gravity.*

Let the fixed line Ox coincide with OA and let Oy be a perpendicular to OA through the point O.

If BL be drawn perpendicular to OA, then $OL=3$ ins., and

$$LB=\sqrt{5^2-3^2}=4 \text{ ins.}$$

Hence, if G be the required centre of gravity and GM be drawn perpendicular to Ox, we have

$$OM=\frac{1.0+2.6+3.3}{1+2+3}=\frac{21}{6}=3\frac{1}{2} \text{ inches,}$$

and
$$MG=\frac{1.0+2.0+3.4}{1+2+3}=\frac{12}{6}=2 \text{ inches.}$$

Hence the required point is obtained by measuring a distance $3\frac{1}{2}$ inches from O along OA and then erecting a perpendicular of length 2 inches.

114. *Centre of Parallel forces.*

The methods and formulae of Arts. 109 and 111 will apply not only to weights, but also to any system of parallel forces and will determine the position of the resultant of any such system. The magnitude of the resultant is the sum of the forces. Each force must, of course, be taken with its proper sign prefixed.

There is one case in which we obtain no satisfactory result; if the algebraic sum of the forces be zero, the resultant force is zero, and the formulae of Art. 111 give

$$\bar{x} = \infty, \text{ and } \bar{y} = \infty.$$

In this case the system of parallel forces is, as in Art. 54, equivalent to a couple.

EXAMPLES. XVI.

1. Particles of 1, 2, 3, and 4 lbs. weight are placed at the angular points of a square; find the distance of their C.G. from the centre of the square.

2. At two opposite corners *A*, *C* of a square *ABCD* weights of 2 lbs. each are placed, and at *B* and *D* are placed 1 and 7 lbs. respectively; find their centre of gravity.

3. Particles of 5, 6, 9 and 7 lbs. respectively are placed at the corners *A*, *B*, *C*, *D* of a horizontal square, the length of whose side is 27 inches; find where a single force must be applied to preserve equilibrium.

4. Five masses of 1, 2, 3, 4, and 5 ounces respectively are placed on a square table. The distances from one edge of the table are 2, 4, 6, 8, and 10 inches and from the adjacent edge 3, 5, 7, 9, and 11 inches respectively. Find the distance of the centre of gravity from the two edges.

5. Weights proportional to 1, 2, and 3 are placed at the corners of an equilateral triangle, whose side is of length *a*; find the distance of their centre of gravity from the first weight.

Find the distance also if the weights be proportional to 11, 13, and 6.

6. *ABC* is an equilateral triangle of side 2 feet. At *A*, *B*, and *C* are placed weights proportional to 5, 1, and 3, and at the middle points of the sides *BC*, *CA*, and *AB* weights proportional to 2, 4, and 6; shew that their centre of gravity is distant 16 inches from *B*.

7. Equal masses, each 1 oz., are placed at the angular points of a heavy triangular lamina, and also at the middle points of its sides; find the position of the centre of gravity of the masses.

8. *ABC* is a triangle right angled at *A*, *AB* being 12 and *AC* 15 inches; weights proportional to 2, 3, 4 respectively are placed at *A*, *C*, and *B*; find the distances of their centre of gravity from *B* and *C*.

9. Particles, of mass 4, 1, and 1 lbs., are placed at the angular points of a triangle; shew that the centre of gravity of the particles bisects the distance between the centre of gravity and one of the vertices of the triangle.

10. Three masses are placed at the angular points of a triangle *ABC*. Find their ratios if their centre of inertia be halfway between *A* and the middle point of *BC*.

11. Bodies of mass 2, 3, and 4 lbs. respectively are placed at the angular points *A*, *B*, *C* of a triangle; find their centre of gravity *G*, and shew that forces $2GA$, $3GB$, $4GC$ are in equilibrium.

12. *ABC* is a uniform triangular plate, of mass 3 lbs. Masses of 2, 3, and 5 lbs. respectively are placed at *A*, *B* and *C*. Find the position of the centre of gravity of the whole system.

13. To the vertices *A*, *B*, *C* of a uniform triangular plate, whose mass is 3 lbs. and whose centre of gravity is *G*, particles of masses 2 lbs., 2 lbs., and 11 lbs., are attached; shew that the centre of gravity of the system is the middle point of *GC*.

14. Masses of 2, 3, 2, 6, 9, and 6 lbs. are placed at the angular corners of a regular hexagon, taken in order; find their centre of gravity.

15. Weights proportional to 5, 4, 6, 2, 7, and 3 are placed at the angular points of a hexagon, taken in order; shew that their centre of gravity is the centre of the hexagon.

16. Weights proportional to 1, 5, 3, 4, 2, and 6 are placed at the angular points of a hexagon, taken in order; shew that their centre of gravity is the centre of the hexagon.

17. If weights proportional to the numbers 1, 2, 3, 4, 5, and 6 be placed at the angular points of a regular hexagon taken in order, shew that the distance of their centre of gravity from the centre of the circumscribing circle of the hexagon is $\frac{2}{7}$ths of the radius of the circle.

18. At the angular points of a square, taken in order, there act parallel forces in the ratio 1 : 3 : 5 : 7; find the distance from the centre of the square of the point at which their resultant acts.

19. *A*, *B*, *C*, *D* are the angles of a parallelogram taken in order; like parallel forces proportional to 6, 10, 14, and 10 respectively act at *A*, *B*, *C* and *D*; shew that the centre and resultant of these parallel forces remain the same, if, instead of these forces, parallel forces, proportional to 8, 12, 16 and 4, act at the points of bisection of the sides *AB*, *BC*, *CD*, and *DA* respectively.

20. Find the centre of parallel forces equal respectively to P, $2P$, $3P$, $4P$, $5P$, and $6P$, the points of application of the forces being at distances 1, 2, 3, 4, 5, and 6 inches respectively from a given point A measured along a given line AB.

21. Three parallel forces P, Q, and R, act at the vertices A, B, and C, of a triangle and are proportional respectively to a, b, and c. Find the magnitude and position of their resultant.

115. *Given the centre of gravity of the two portions of a body, to find the centre of gravity of the whole body.*

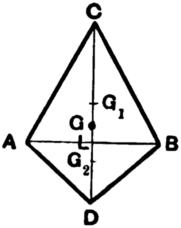
Let the given centres of gravity be G_1 and G_2, and let the weights of the two portions be W_1 and W_2; the required point G, by Art. 97, divides G_1G_2 so that

$$G_1G : GG_2 :: W_2 : W_1.$$

The point G may also be obtained by the use of Art. 109.

Ex. *On the same base AB, and on opposite sides of it, isosceles triangles CAB, DAB are described whose altitudes are* 12 *inches and* 6 *inches respectively. Find the distance from AB of the centre of gravity of the quadrilateral $CADB$.*

Let CLD be the perpendicular to AB, meeting it in L, and let



G_1, G_2 be the centres of gravity of the two triangles CAB, DAB respectively. Hence

$$CG_1 = \tfrac{2}{3}\,.\,CL = 8, \text{ and } CG_2 = CL + LG_2 = 12 + 2 = 14.$$

The weights of the triangles are proportional to their areas, *i.e.*, to $\tfrac{1}{2}AB\,.\,12$ and $\tfrac{1}{2}AB\,.\,6$.

If G be the centre of gravity of the whole figure, we have

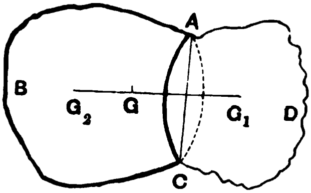
$$CG=\frac{\triangle CAB\times CG_1+\triangle DAB\times CG_2}{\triangle CAB+\triangle DAB}$$

$$=\frac{\frac{1}{2}AB\,.\,12\times 8+\frac{1}{2}AB\,.\,6\times 14}{\frac{1}{2}AB\,.\,12+\frac{1}{2}AB\,.\,6}=\frac{48+42}{6+3}=\frac{90}{9}=10.$$

Hence $LG=CL-CG=2$ inches.

116. *Given the centre of gravity of the whole of a body and of a portion of the body, to find the centre of gravity of the remainder.*

Let G be the centre of gravity of a body $ABCD$, and G_1 that of the portion ADC.



Let W be the weight of the whole body and W_1 that of the portion ACD, so that $W_2 (= W - W_1)$ is the weight of the portion ABC.

Let G_2 be the centre of gravity of the portion ABC. Since the two portions of the body make up the whole, therefore W_1 at G_1 and W_2 at G_2 must have their centre of gravity at G.

Hence G must lie on G_1G_2 and be such that

$$W_1\,.\,GG_1=W_2\,.\,GG_2.$$

Hence, given G and G_1, we obtain G_2 by producing G_1G to G_2, so that

$$GG_2=\frac{W_1}{W_2}\,.\,GG_1$$

$$=\frac{W_1}{W-W_1}\,.\,GG_1.$$

The required point may be also obtained by means of Art. 109.

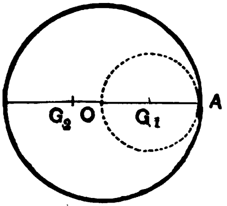
Ex. 1. *From a circular disc, of radius r, is cut out a circle, whose diameter is a radius of the disc; find the centre of gravity of the remainder.*

Since the areas of circles are to one another as the squares of their radii,

∴ area of the portion cut out : area of the whole circle

$$:: \left(\frac{r}{2}\right)^2 : r^2$$

$$:: 1 : 4.$$



Hence the portion cut off is one-quarter, and the portion remaining is three-quarters, of the whole, so that $W_1 = \frac{1}{3} W_2$.

Now the portions W_1 and W_2 make up the whole disc, and therefore balance about O.

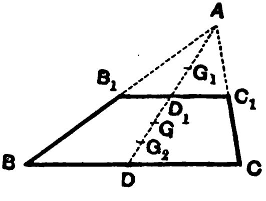
Hence $$W_2 \cdot OG_2 = W_1 \cdot OG_1 = \tfrac{1}{3} W_2 \times \tfrac{1}{2} r.$$

$$\therefore\ OG_2 = \tfrac{1}{6} r.$$

Ex. 2. *From a triangular lamina ABC is cut off, by a line parallel to its base BC, one-quarter of its area; find the centre of gravity of the remainder.*

Let AB_1C_1 be the portion cut off, so that

$$\triangle AB_1C_1 : \triangle ABC :: 1 : 4.$$



By Euc. VI. 19, since the triangles AB_1C_1, ABC, are similar, we have

$$\triangle AB_1C_1 : \triangle ABC :: AB_1^2 : AB^2.$$

$$\therefore AB_1^2 : AB^2 :: 1 : 4,$$

and hence $$AB_1 = \tfrac{1}{2}AB.$$

The line B_1C_1 therefore bisects AB, AC, and AD.

Let G, G_1 be the centres of gravity of the triangles ABC and AB_1C_1 respectively; also let W_1, W_2 be the respective weights of the portion cut off and the portion remaining, so that $W_2 = 3W_1$.

Since W_2 at G_2 and W_1 at G_1 balance about G, we have, by Art. 109,

$$DG = \frac{W_1 . DG_1 + W_2 . DG_2}{W_1 + W_2} = \frac{DG_1 + 3DG_2}{4} \quad \text{...............(i).}$$

But $$DG = \tfrac{1}{3}DA = \tfrac{2}{3}DD_1;$$

and $$DG_1 = DD_1 + \tfrac{1}{3}D_1A = DD_1 + \tfrac{1}{3}DD_1 = \tfrac{4}{3}DD_1.$$

Hence (i) is $$4 \times \tfrac{2}{3}DD_1 = \tfrac{4}{3}DD_1 + 3DG_2.$$

$$\therefore DG_2 = \tfrac{4}{9}DD_1.$$

EXAMPLES. XVII.

1. A uniform rod, 1 foot in length, is broken into two parts, of lengths 5 and 7 inches, which are placed so as to form the letter T, the longer portion being vertical; find the centre of gravity of the system.

2. Two rectangular pieces of the same cardboard, of lengths 6 and 8 inches and breadths 2 and 2½ inches respectively, are placed touching, but not overlapping, one another on a table so as to form a T-shaped figure. Find the position of its centre of gravity.

3. A heavy beam consists of two portions, whose lengths are as 3 : 5, and whose weights are as 3 : 1; find the position of its centre of gravity.

4. Two sides of a rectangle are double of the other two, and on one of the longer sides an equilateral triangle is described; find the centre of gravity of the lamina made up of the rectangle and the triangle.

5. A piece of cardboard is in the shape of a square $ABCD$ with an isosceles triangle described on the side BC; if the side of the square be 12 inches and the height of the triangle be 6 inches, find the distance of the centre of gravity of the cardboard from the line AD.

6. An isosceles right-angled triangle has squares described externally on all its sides. Shew that the centre of gravity of the figure so formed is on the line which bisects the hypothenuse, passes through the right angle, and divides it in the ratio 1 : 26.

7. Two uniform spheres, composed of the same materials, and whose diameters are 6 and 12 inches respectively, are firmly united; find the position of their centre of gravity.

8. From a parallelogram is cut one of the four portions into which it is divided by its diagonals; find the centre of gravity of the remainder.

9. A parallelogram is divided into four parts, by joining the middle points of opposite sides, and one part is cut away; find the centre of gravity of the remainder.

10. From a square a triangular portion is cut off, by cutting the square along a line joining the middle points of two adjacent sides; find the centre of gravity of the remainder.

11. From a triangle is cut off $\frac{1}{9}$th of its area by a straight line parallel to its base. Find the position of the centre of gravity of the remainder.

12. ABC is an equilateral triangle, of 6 inches side, of which O is the centre of gravity. If the triangle OBC be removed, find the centre of gravity of the remainder.

13. If from a triangle ABC three equal triangles ARQ, BPR, and CQP, be cut off, shew that the centres of inertia of the triangles ABC and PQR are coincident.

14. G is the centre of gravity of a given isosceles triangle, right angled at A, and having BC equal to a. The portion GBC is cut away; find the distance of the centre of gravity of the remainder from A.

15. On the same base BC are two triangles, ABC and $A'BC$, the vextex A' falling within the former triangle. Find the position of A' when it is the centre of gravity of the area between the two triangles.

16. Two triangles, each $\frac{1}{m}$th of the whole, are cut off from a given triangle at two of its angular points, B and C, by straight lines parallel to the opposite sides; find the C.G. of the remainder.

17. Out of a square plate shew how to cut a triangle, having one side of the square for base, so that the remainder may have its centre of gravity at the vertex of this triangle.

18. A uniform plate of metal, 10 inches square, has a hole of area 3 square inches cut out of it, the centre of the hole being $2\frac{1}{2}$ inches from the centre of the plate; find the position of the centre of gravity of the remainder of the plate.

19. Where must a circular hole, of 1 foot radius, be punched out of a circular disc, of 3 feet radius, so that the centre of gravity of the remainder may be 2 inches from the centre of the disc?

20. Two spheres, of radii a and b, touch internally; find the centre of gravity of the solid included between them.

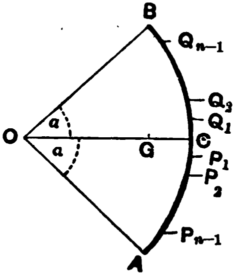
21. If a right cone be cut by a plane bisecting its axis, find the distance of the vertex of the cone from the centre of gravity of the frustum thus cut off.

22. A solid right circular cone of homogeneous iron, of height 64 inches and mass 8192 lbs., is cut by a plane perpendicular to its axis so that the mass of the small cone removed is 686 lbs. Find the height of the centre of gravity of the truncated portion above the base of the cone.

23. A solid right circular cone has its base scooped out, so that the hollow is a right cone on the same base; how much must be scooped out so that the centre of gravity of the remainder may coincide with the vertex of the hollow?

*117. Centre of gravity of a uniform circular arc.

Let AB be a circular arc, subtending an angle 2α at its centre O, and let OC bisect the angle AOB.



Let the arc AB be divided into $2n$ equal portions, the points of division, starting from C, being P_1, P_2, ... P_{n-1} towards A, and Q_1, Q_2, ... Q_{n-1} towards B.

At each of these points of division, and at the extremities A, B, and also at the point C, let there be placed equal particles, each of mass m.

Let the arc joining two successive particles subtend an angle β at the centre O, so that $2n\beta = 2\alpha$.

Since the system of particles is symmetrical with respect to the line OC, the centre of gravity, G, must lie on the line OC. Let $\bar{x}$ be the distance OG.

Then, by Art. 111,

$$\bar{x} = \frac{mr + 2m \,.\, r\cos\beta + 2m \,.\, r\cos 2\beta + \ldots + 2mr\cos n\beta}{m + 2m + 2m + \ldots + 2m}$$

$$= \frac{r}{2n+1}[1 + 2\cos\beta + 2\cos 2\beta + \ldots + 2\cos n\beta]$$

$$= \frac{r}{2n+1}\left[1 + 2\frac{\cos\frac{n+1}{2}\beta\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$ (Todhunter's *Trigonometry*, Art. 304.)

$$= \frac{r}{2n+1}\left[1 + \frac{\sin\left(n+\frac{1}{2}\right)\beta - \sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$= r\frac{\sin\left(n+\frac{1}{2}\right)\beta}{(2n+1)\sin\frac{\beta}{2}} = r\frac{\sin\left(a+\frac{a}{2n}\right)}{(2n+1)\sin\frac{a}{2n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Now let the number of particles be increased without limit, a remaining constant, and consequently β decreasing without limit. We thus obtain the case of a uniform circular arc.

Now $$(2n+1)\sin\frac{a}{2n} = \frac{(2n+1)}{2n}a \,.\, \frac{\sin\frac{a}{2n}}{\frac{a}{2n}}$$

$$= \left[1 + \frac{1}{2n}\right] . \, a \,.\, \frac{\sin\frac{a}{2n}}{\frac{a}{2n}} = a,$$

when n is made indefinitely great.

Hence, in the case of a uniform circular arc, (i) becomes

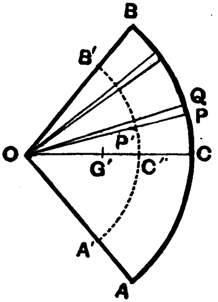
$$\bar{x} = r\frac{\sin a}{a}.$$

Cor. In the case of a semicircular arc, in which $\alpha = \frac{\pi}{2}$, the distance of the centre of gravity from the centre

$$= r\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2r}{\pi}.$$

*118. Centre of gravity of a sector of a circle.

With the same notation as in the last article, let P and Q be two consecutive points on the circular boundary of the sector, so that PQ is very approximately a straight line, and OPQ is a triangle with a very small vertical angle at O.



Take P' on OP such that $OP' = \frac{2}{3}OP$; when PQ is very small, P' is the centre of gravity of the triangle OPQ.

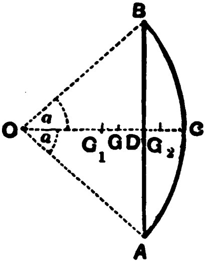
By joining O to an indefinitely large number of consecutive points on the arc AB, the sector can be divided into an indefinitely large number of triangles, each of whose centres of gravity lies on the dotted circular arc, whose radius is $\frac{2}{3}r$.

Hence the centre of gravity of the sector is the same as that of the circular arc $A'C'B'$, so that, by the last article,

$$OG' = OC'\frac{\sin a}{a} = \tfrac{2}{3}r\frac{\sin a}{a}.$$

*119. Centre of gravity of the segment of a circle.

The segment of a circle ACB is the difference between the sector $OACB$ and the triangle OAB.



Using the same notation as in the two previous articles, let G_1, G_2 be respectively the centres of gravity of the triangle AOB and the segment ACB. Also let G be the centre of gravity of the sector, and let AB meet OC in D.

We have, by Art. 109,

$$OG = \frac{\triangle AOB \times OG_1 + \text{segment } ACB \times OG_2}{\triangle AOB + \text{segment } ACB} \quad \text{......(i).}$$

But $$OG_1 = \tfrac{2}{3}OD = \tfrac{2}{3}r\cos a,$$

and $$OG = \tfrac{2}{3}r\frac{\sin a}{a}.$$

Also $$\triangle AOB = \tfrac{1}{2}r^2\sin 2a,$$

and $$\text{segment } ABC = \text{sector } AOB - \triangle AOB$$
$$= \tfrac{1}{2}r^2 . 2a - \tfrac{1}{2}r^2\sin 2a.$$

Hence equation (i) becomes

$$\tfrac{2}{3}r\frac{\sin\alpha}{\alpha}=\frac{\frac{1}{2}r^2\sin 2\alpha\times\frac{2}{3}r\cos\alpha+\frac{1}{2}r^2(2\alpha-\sin 2\alpha)\times OG_2}{\frac{1}{2}r^2\,.\,2\alpha}$$

$$=\frac{\frac{2}{3}r\cos\alpha\sin 2\alpha+OG_2(2\alpha-\sin 2\alpha)}{2\alpha};$$

$$\therefore\ \tfrac{4}{3}r\sin\alpha-\tfrac{2}{3}r\cos\alpha\sin 2\alpha=OG_2(2\alpha-\sin 2\alpha);$$

$$\therefore\ OG_2=\tfrac{4}{3}r\,\frac{\sin\alpha-\cos^2\alpha\sin\alpha}{2\alpha-\sin 2\alpha}$$

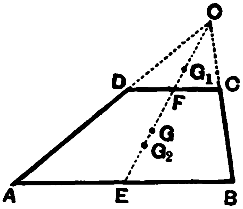
$$=\tfrac{4}{3}r\,\frac{\sin^3\alpha}{2\alpha-\sin 2\alpha}.$$

*120. Centre of gravity of a hemisphere.

If a hemisphere be of radius r, the centre of gravity lies on that radius which is perpendicular to its plane face, and is at a distance $\frac{3r}{8}$ from the centre of the plane face. If the hemisphere be hollow, the distance is $\frac{r}{2}$. The proofs of these statements are difficult by elementary methods, and we shall not give them here.

*121. *To find the centre of gravity of a quadrilateral lamina having two parallel sides.*

Let $ABCD$ be the quadrilateral, having the sides AB and CD parallel and equal to $2a$ and $2b$ respectively.



Let AD and BC be produced to meet in O. Join O to the middle point, F, of CD and produce OF to meet

AB in *E*. Since *AB* and *CD* are parallel, the point *E* will be the middle point of *AB*. Let *EF* be h.

Since *ABCD* is the difference between the triangles *OAB* and *ODC*, we see that its centre of gravity must lie on *EF*.

By similar triangles *OFC*, *OEB*, we have

$$\frac{OF}{b}=\frac{OE}{a}=\frac{EF}{a-b}=\frac{h}{a-b}\ldots\ldots\ldots\ldots\ldots(1).$$

Also, by Euc. VI. 19,

$$\frac{\triangle\, OCD}{b^2}=\frac{\triangle\, OAB}{a^2}=\frac{\text{quad. } ABCD}{a^2-b^2}\ldots\ldots\ldots(2).$$

Let G_1, G_2, G be respectively the centres of gravity of the △ *OCD*, the quadrilateral *ABCD*, and the △ *OAB* respectively.

By Art. 109,

$$EG=\frac{\triangle\, OCD\times EG_1+\text{quad. } ABCD\times EG_2}{\triangle\, OCD+\text{quad. } ABCD}$$

$$=\frac{b^2\,.\,EG_1+(a^2-b^2)\,EG_2}{a^2},\text{ by equation (2);}$$

$$\therefore\ (a^2-b^2)\,EG_2=a^2\,.\,EG-b^2\,.\,EG_1$$

$$=a^2\,.\,\tfrac{1}{3}EO-b^2\left(EF+\tfrac{1}{3}FO\right)$$

$$=a^2\,.\,\frac{1}{3}\,\frac{a}{a-b}\,h-b^2\left[h+\frac{1}{3}\,\frac{b}{a-b}\,h\right],\text{ by (1)}$$

$$=h\left[\frac{1}{3}\,\frac{a^3-b^3}{a-b}-b^2\right]=\frac{h}{3}\left[a^2+ab-2b^2\right].$$

$$\therefore\ EG_2=\frac{h}{3}\,\frac{a^2+ab-2b^2}{a^2-b^2}=\frac{h}{3}\,\frac{a+2b}{a+b}.$$

Hence $$FG_2=h-EG_2=\frac{h}{3}\,\frac{2a+b}{a+b}.$$

$$\therefore\ EG_2\ :\ FG_2\ ::\ a+2b\ :\ 2a+b.$$

EXAMPLES. XVIII.

1. A triangular table rests on supports at its vertices; weights of 6, 8, and 10 lbs. are placed at the middle points of the sides. Find by how much the pressures on the legs are increased thereby.

2. A piece of thin uniform wire is bent into the form of a four-sided figure, $ABCD$, of which the sides AB and CD are parallel, and BC and DA are equally inclined to AB. If AB be 18 inches, CD 12 inches, and BC and DA each 5 inches, find the distance from AB of the centre of gravity of the wire.

3. AB, BC, CD are three equal uniform rods firmly joined, so as to form three successive sides of a regular hexagon, and are suspended from the point A; shew that CD is horizontal.

4. ABC is a piece of uniform wire; its two parts AB, BC are straight, and the angle ABC is 135°. It is suspended from a fixed point by a string attached to the wire at B, and the part AB is observed to be horizontal. Shew that BC is to AC as $\sqrt[4]{2}$ to 1.

5. A rod, of length $5a$, is bent so as to form 5 sides of a regular hexagon; shew that the distance of its centre of gravity from either end of the wire is $\frac{a}{10}\sqrt{133}$.

6. The side CD of a uniform trapezoidal lamina $ABCD$ is twice as long as AB, to which it is opposite and parallel; compare the distances of the centres of gravity of $ABCD$ from AB and CD.

7. If the centre of gravity of a quadrilateral lamina $ABCD$ coincide with one of the angles A, shew that the distances of A and C from the line BD are as $1 : 2$.

8. A uniform quadrilateral $ABCD$ has the sides AB, AD, and the diagonal AC all equal, and the angles BAC and CAD are 30° and 60° respectively. If a weight, equal to two-thirds that of the triangle ABC, be attached at the point B, and the whole rest suspended from the point A, shew that the diagonal AC will be vertical.

9. Explain what will take place when 3 forces, represented by AB, BC, and CA respectively, act along the side of a triangular board ABC which is supported on a smooth peg passing through its centre of gravity.

10. Three forces act at a point O in the plane of a triangle ABC, being represented by OA, OB, OC; where must be the point O so that the three forces may be in equilibrium?

11. A particle P is attracted to three points A, B, C by forces equal to $\mu . PA$, $\mu . PB$, and $\mu . PC$ respectively; shew that the resultant is $3\mu . PG$, where G is the centre of gravity of the triangle ABC.

12. A particle P is acted upon by forces towards the points $A, B, C,\ldots$ which are represented by $\lambda . PA$, $\mu . PB$, $\nu . PC$, ...; shew that their resultant is represented by $(\lambda+\mu+\nu+\ldots) PG$, where G is the centre of gravity of weights placed at A, B, $C\ldots$ proportional to λ, μ, $\nu\ldots$ respectively.

[This is the generalised form of Art. 42, and may be proved by successive applications of that article.]

13. A rod is hung up by two strings attached to its ends; shew that the tensions of the strings are proportional to their lengths.

Prove that the same relation holds for a triangular lamina hung up by three strings attached to its angular points.

14. Find the vertical angle of a cone in order that the centre of gravity of its whole surface may coincide with the centre of gravity of its volume.

15. A cylinder and a cone have their bases joined together, the bases being of the same size; find the ratio of the height of the cone to the height of the cylinder so that the common centre of gravity may be at the centre of the common base.

16. Shew how to cut out of a uniform cylinder a cone, whose base coincides with that of the cylinder, so that the centre of gravity of the remaining solid may coincide with the vertex of the cone.

17. If the diameter of the base of a cone be to its altitude as $1 : \sqrt{2}$, shew that, when the greatest possible sphere has been cut out, the centre of gravity of the remainder coincides with that of the cone.

18. From a uniform right cone, whose vertical angle is 60°, is cut out the greatest possible sphere; shew that the centre of gravity of the remainder divides the axis in the ratio 11 : 49.

19. A solid in the form of a right circular cone has its base scooped out, so that the hollow so formed is a right circular cone on the same base and of half the height of the original cone; find the position of the centre of gravity of the cone so formed.

20. A uniform equilateral triangle ABC is supported with the angle A in contact with a smooth wall by means of a string BD, equal in length to a side of the triangle, which is fastened to a point D vertically above A. Shew that the distances of B and C from the wall are as 1 : 5.

21. A cone, whose height is equal to four times the radius of its base, is hung from a point in the circumference of its base; shew that it will rest with its base and axis equally inclined to the vertical.

22. Two right cones, consisting of the same material, have equal slant slides and vertical angles of 60° and 120° respectively, and are so joined that they have a slant side coincident. Shew that, if they be suspended from their common vertex, the line of contact will be inclined at 15° to the vertical.

23. A triangular piece of paper is folded across the line bisecting two sides, the vertex being thus brought to lie on the base of the triangle. Shew that the distance of the centre of inertia of the paper in this position from the base of the triangle is three-quarters that of the centre of inertia of the unfolded paper from the same line.

24. A rectangular sheet of stiff paper, whose length is to its breadth as $\sqrt{2}$ to 1, lies on a horizontal table with its longer sides perpendicular to the edge and projecting over it. The corners of the paper are doubled over symmetrically, so that the creases pass through the middle point of the side joining the corners and make angles of 45° with it. The paper is then on the point of falling over; shew that it had originally $\frac{23}{48}$ths of its length on the table.

25. At each of $n-1$ of the angular points of a regular polygon of n sides a particle is placed, the particles being equal; shew that the distance of their centre of gravity from the centre of the circle circumscribing the polygon is $\frac{r}{n-1}$, where r is the radius of the circle.

26. A square hole is punched out of a circular lamina, the diagonal of the square being a radius of the circle. Shew that the centre of gravity of the remainder is at a distance $\frac{a}{8\pi-4}$ from the centre of the circle, where a is the diameter of the circle.

27. From a uniform triangular board the portion consisting of the area of the inscribed circle is removed; shew that the distance of the centre of gravity of the remainder from any side, a, is

$$\frac{S}{3as}\cdot\frac{2s^3-3\pi aS}{s^2-\pi S},$$

where S is the area and s the semiperimeter of the board.

28. A circular hole of a given size is punched out of a uniform circular plate; shew that the centre of gravity lies within a certain circle.

29. The distances of the angular points and intersection of the diagonals of a plane quadrilateral lamina from any line in its plane are a, b, c, d, and e; shew that the distance of the centre of inertia from the same line is $\frac{1}{3}(a+b+c+d-e)$.

30. If A and B be the positions of two masses, m and n, and if G be their centre of gravity, shew that, if P be any point, then

$$m\,.\,AP^2+n\,.\,BP^2=m\,.\,AG^2+n\,.\,BG^2+(m+n)\,PG^2.$$

Similarly, if there be any number of masses m, n, p, ... at points A, B, C, ..., and G be their centre of gravity, shew that

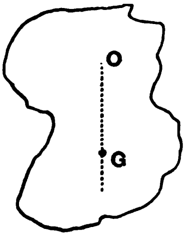
$$\begin{aligned}&m\,.\,AP^2+n\,.\,BP^2+p\,.\,CP^2+\dots\\&\qquad=m\,.\,AG^2+n\,.\,BG^2+p\,.\,CG^2+\dots+(m+n+p+\dots)\,PG^2.\end{aligned}$$

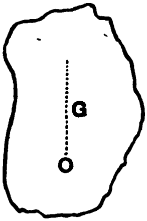
CHAPTER X.

CENTRE OF GRAVITY (CONTINUED).

122. *If a rigid body be in equilibrium, one point only of the body being fixed, the centre of gravity of the body will be in the vertical line passing through the fixed point of the body.*

Let O be the fixed point of the body, and G its centre of gravity.





The forces acting on the body are the reaction at the fixed point of support of the body, and the weights of the component parts of the body.

The weights of these component parts are equivalent to a single vertical force through the centre of gravity of the body.

Also, when two forces keep a body in equilibrium, they must be equal and opposite and have the same line of action. But the lines of action cannot be the same unless the vertical line through *G* passes through the point *O*.

Two cases arise, the first, in which the centre of gravity *G* is below the point of suspension *O*, and the second, in which *G* is above *O*.

In the first case, the body, if slightly displaced from its position of equilibrium, will tend to return to this position; in the second case, the body will not tend to return to its position of equilibrium.

123. *To find, by experiment, the centre of gravity of a body of any shape.*

Fix one point *O* of the body and let the body assume its position of equilibrium. Take a point *A* of the body vertically below *O*; then, by Art. 122, the centre of gravity is somewhere in the line *OA*.

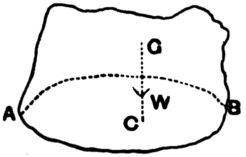
Secondly, release the point *O* of the body, and fix a second point *O′* (not in the straight line *OA*); let the body take up its new position of equilibrium. Take a point *A′* in the body, vertically below *O′*, so that the centre of gravity is somewhere in the line *O′A′*.

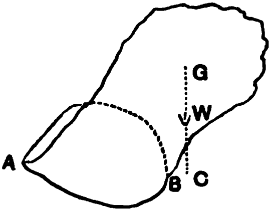
The required centre of gravity will therefore be the point of intersection of the lines *OA* and *O′A′*.

The student should apply this method in the case of a body such as an irregularly shaped piece of paper; the points *O* and *O′* can be easily supported by means of a pin put through the paper.

124. *If a body be placed with its base in contact with a horizontal plane, it will stand, or fall, according as the vertical line drawn through the centre of gravity of the body meets the plane within, or without, the base.*

The forces acting on the body are its weight, which acts at its centre of gravity G, and the reactions of the plane,



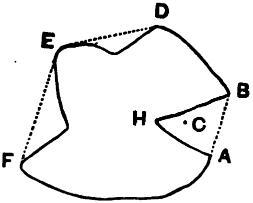


acting at different points of the base of the body. These reactions are all vertical, and hence they may be compounded into a single vertical force acting at some point of the base.

Since the resultant of two *like* parallel forces acts always at a point *between* the forces, it follows that the resultant of all the pressures on the base of the body cannot act through a point *outside* the base.

Hence, if the vertical line through the centre of gravity of the body meet the plane at a point outside the base, it cannot be balanced by the resultant pressure, and the body cannot therefore be in equilibrium, but must fall over.

If the base of the body be a figure having a re-entrant



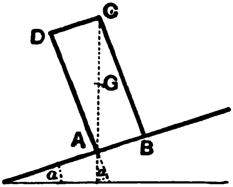
angle, as in the above figure, we must extend the meaning of the word "base" in the enunciation to mean the area included in the figure obtained by drawing a piece of thread tightly round the geometrical base. In the above figure the "base" therefore means the area *ABDEFA*.

For example, the point *C*, at which the resultant pressure acts, may lie within the area *AHB*, but it cannot lie without the dotted line *AB*.

If the point *C* were on the line *AB*, between *A* and *B*, the body would be on the point of falling over.

125. **Ex.** *A cylinder, of height* h, *and the radius of whose base is* r, *is placed on an inclined plane and prevented from sliding; if the inclination of the plane be gradually increased, find when the cylinder will topple.*

Let the figure represent the section of the cylinder when it is on the point of toppling over; the vertical line through the centre of gravity *G* of the body must therefore just pass through the end *A* of the base. Hence *CAD* must be equal to the angle of inclination, a, of the plane.



Hence $$\frac{h}{2r}=\frac{CB}{AB}=\tan CAB=\cot a;$$

$$\therefore \tan a=\frac{2r}{h},$$

giving the required inclination of the plane.

Stable, unstable, and neutral equilibrium.

126. We have pointed out in Art. 122 that the body in the first figure of that article would, if slightly displaced, tend to return to its position of equilibrium, and that the body in the second figure would not tend to return

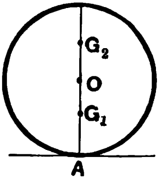
to its original position of equilibrium, but would recede still further from that position.

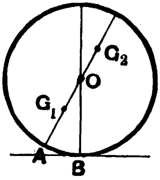
These two bodies are said to be in stable and unstable equilibrium respectively.

Again, a cone, resting with its flat circular base in contact with a horizontal plane, would, if slightly displaced, return to its position of equilibrium; if resting with its vertex in contact with the plane it would, if slightly displaced, recede still further from its position of equilibrium; whilst, if placed with its slant side in contact with the plane, it will remain in equilibrium in any position. The equilibrium in the latter case is said to be neutral.

127. Consider, again, the case of a heavy sphere, resting on a horizontal plane, whose centre of gravity is not at its centre.

Let the first figure represent the position of equilibrium, the centre of gravity being either below the centre O, as G_1, or above, as G_2. Let the second figure represent the sphere turned through a small angle, so that B is now the point of contact with the plane.





The pressure of the plane still acts through the centre of the sphere.

If the weight of the body act through G_1, it is clear that the body will return towards its original position of equilibrium, and the body was in stable equilibrium.

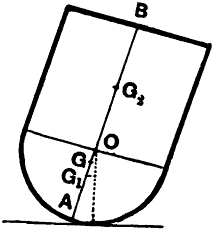
If the weight act through G_2, the body will move still further from its original position of equilibrium, and it was in unstable equilibrium.

If however the centre of gravity of the body had been at O, then, in the case of the second figure, the weight would still be balanced by the pressure of the plane; the body would thus remain in the new position, and the equilibrium would be called neutral.

128. **Def.** A body is said to be in **stable** equilibrium when, if it be slightly displaced from its position of equilibrium, the forces acting on the body tend to make it return towards its position of equilibrium; it is in **unstable** equilibrium when, if it be slightly displaced, the forces tend to move it still further from its position of equilibrium; it is in **neutral** equilibrium, if the forces acting on it in its displaced position be in equilibrium.

129. **Ex.** *A homogeneous body, consisting of a cylinder and a hemisphere joined at their bases, is placed with the hemispherical end on a horizontal table; is the equilibrium stable or unstable?*

Let G_1 and G_2 be the centres of gravity of the hemisphere and



cylinder, and let A be the point of the compound body initially in contact with the table, and let O be the centre of the base of the hemisphere.

If h be the height of the cylinder, and r be the radius of the base, we have

$$OG_1 = \tfrac{3}{8}r \text{ (Art. 120), and } OG_2 = \frac{h}{2}.$$

Also the weights of the hemisphere and cylinder are proportional to $\frac{2}{3}\pi r^3$ and $\pi . r^2 h$.

The reaction of the plane, in the displaced position of the body, always passes through the centre O.

The equilibrium is stable or unstable according as G, the centre of gravity of the compound body, is below or above O,

i. e., according as

$$OG_1 \times \text{wt. of hemisphere is} \gtrless OG_2 \times \text{wt. of cylinder,}$$

i.e., according as $\frac{3}{8}r \times \frac{2}{3}\pi r^3 \text{ is} \gtrless \frac{h}{2} \times \pi r^2 h,$

i.e., according as $\frac{r^2}{2} \text{ is} \gtrless h^2,$

i.e., according as $r \text{ is} \gtrless \sqrt{2}h.$

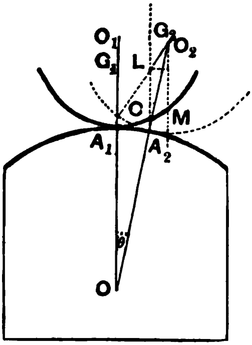
**130. Within the limits of this book we cannot enter into the general discussion of the equilibrium of one body resting on another; in the following article we shall discuss the case in which the portions of the two bodies in contact are spherical.

A body rests in equilibrium upon another fixed body, the portions of the two bodies in contact being spheres of radii r and R respectively; if the first body be slightly displaced to find whether the equilibrium is stable or unstable.

Let O be the centre of the spherical surface of the lower body, and O_1 that of the upper body; since there is equilibrium, the centre of gravity G_1 of the upper body must be in the line OO_1, which passes through the point of contact A_1 of the bodies.

Let A_1G_1 be h.

Let the upper body be slightly displaced, by rolling, so that the new position of the centre of the upper body



is O_2, the new point of contact is A_2, the new position of the centre of gravity is G_2, and the new position of the point A_1 is C. Hence CG_2 is h.

Through A_2 draw A_2L vertically to meet O_2C in L, and draw O_2M vertically downwards.

Let the angle A_2OA_1 be θ, and let A_2O_2C be ϕ, so that the angle CO_2M is $(\theta + \phi)$.

Since the upper body has rolled into its new position, the arc A_1A_2 is equal to the arc CA_2.

Hence $$R \,.\, \theta = r \,.\, \phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
where r and R are respectively the radii of the upper and lower surfaces.

The equilibrium is stable, or unstable, according as G_2 lies to the left, or right, of the line A_2L,

i.e., according as the distance of G_2 from O_2M is

$$> \text{ or } < \text{ the distance of } L \text{ from } O_2M,$$

i.e., according as

$$O_2G_2 \sin(\theta + \phi) \text{ is } > \text{ or } < O_2A_2 \sin\theta,$$

i.e., according as

$$(r-h)\sin(\theta+\phi) \text{ is } > \text{ or } < r\sin\theta,$$

i.e., according as

$$\frac{r-h}{r} \text{ is } > \text{ or } < \frac{\sin\theta}{\sin(\theta+\phi)}.$$

But $$\frac{\sin\theta}{\sin(\theta+\phi)} = \frac{\theta}{\theta+\phi},$$

since θ and ϕ are both very small,

$$= \frac{r}{r+R}, \text{ by equation (1).}$$

Hence the equilibrium is stable, or unstable, according as

$$\frac{r-h}{r} \text{ is } > \text{ or } < \frac{r}{r+R},$$

i.e., according as $r - \frac{r^2}{r+R}$ is $>$ or $< h$,

i.e., according as $\frac{Rr}{r+R}$ is $>$ or $< h$,

i.e., according as

$$\frac{1}{h} \text{ is } > \text{ or } < \frac{1}{r} + \frac{1}{R}.$$

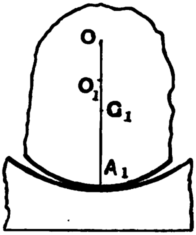
If $\frac{1}{h} = \frac{1}{r} + \frac{1}{R}$, the equilibrium is sometimes said to be neutral; it is however really unstable, but the investigation is beyond the limits of this book.

Hence the equilibrium is stable only when

$$\frac{1}{h} \text{ is } > \frac{1}{r} + \frac{1}{R};$$

in all other cases it is unstable.

Cor. 1. If the surface of the lower body, instead of being convex, as in the above figure, be concave, as in the following figure, the above investigation will still apply provided we change the sign of R.



Hence the equilibrium is stable when

$$\frac{1}{h} \text{ is } > \frac{1}{r} - \frac{1}{R};$$

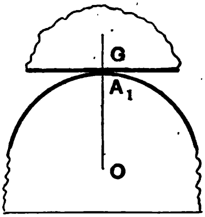
otherwise it is unstable.

Cor. 2. If the upper body have a plane face in contact with the lower body, as in the following figure, r is now infinite in value, and therefore $\frac{1}{r}$ is zero.

Hence the equilibrium is stable if

$$\frac{1}{h} \text{ is } > \frac{1}{R};$$

i.e., if h is $< R$.



Hence the equilibrium is stable, if the distance of the centre of gravity of the upper body from its plane face be less than the radius of the lower body; otherwise the equilibrium is unstable.

Cor. 3. If the lower body be a plane, so that R is infinity, the equilibrium is stable if

$$\frac{1}{h} \text{ be} > \frac{1}{r}, \textit{ i.e.}, \text{ if } h \text{ be} < r.$$

Hence, if a body of spherical base be placed on a horizontal table, it is in stable equilibrium, if the distance of its centre of gravity from the point of contact be less than the radius of the spherical surface.

EXAMPLES. XIX.

1. A carpenter's rule, 2 feet in length, is bent into two parts at right angles to one another, the length of the shorter portion being 8 inches. If the shorter be placed on a smooth horizontal table, what is the length of the least portion on the table that there may be equilibrium?

2. A piece of metal, 18 cubic inches in volume, is made into a cylinder which rests with its base on an inclined plane, of 30° slope, and is prevented from slipping. How tall may the cylinder be made so that it may just not topple over?

3. If a triangular lamina can just stand on its edge on a smooth table, shew that the following relation must hold between the sides;

$$a^2 = b^2 + 3c^2.$$

4. The side CD of a uniform square plate $ABCD$, whose weight is W, is bisected at E and the triangle AED is cut off. The plate $ABCEA$ is placed in a vertical position with the side CE on a horizontal plane. What is the greatest weight that can be placed at A without upsetting the plate?

5. ABC is a flat board, A being a right angle and AC in contact with a flat table; D is the middle point of AC and the triangle ABD is cut away; shew that the triangle is just on the point of falling over.

6. A brick is laid with one-quarter of its length projecting over the edge of a wall; a brick and one-quarter of a brick are then laid on the first with one-quarter of a brick projecting over the edge of the first brick; a brick and a half are laid on this, and so on; shew that 4 courses of brick laid in the above manner will be in equilibrium without the aid of mortar, but that, if a fifth course be added, the structure will topple.

7. How many coins, of the same size and having their thicknesses equal to $\frac{1}{10}$th of their diameters, can stand in a cylindrical pile on an inclined plane, whose height is one-sixth of the base, assuming that there is no slipping?

If the edge of each coin overlap on one side that of the coin below, find by what fraction of the diameter each must overlap so that a pile of unlimited height may stand on the plane.

8. A number of bricks, each 9 inches long, 4 inches wide, and 3 inches thick, are placed one on another so that, whilst their narrowest surfaces, or thicknesses, are in the same vertical plane, each brick overlaps the one underneath it by half an inch; the lowest brick being placed on a table, how many bricks can be so placed without their falling over?

9. ABC is an isosceles triangle, of weight W, of which the angle A is 120°, and the side AB rests on a smooth horizontal table, the plane of the triangle being vertical; if a weight $\frac{W}{3}$ be hung on at C, shew that the triangle will just be on the point of toppling over.

10. The quadrilateral lamina $ABCD$ is formed of two uniform isosceles triangles ABC and ADC, whose vertices are B and D, on opposite sides of a common base AC, the angle ABC being a right angle. Shew that it will rest in a vertical plane with BC on a horizontal plane, provided the area of ADC be not greater than four times that of ABC.

11. A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table, the hemisphere being in contact with the table; find the greatest height of the cone so that the equilibrium may be stable.

12. A solid consists of a cylinder and a hemisphere of equal radius, fixed base to base; find the ratio of the height to the radius of the cylinder, so that the equilibrium may be neutral when the spherical surface rests on a horizontal plane.

13. A hemisphere rests in equilibrium on a sphere of equal radius; shew that the equilibrium is unstable when the curved, and stable when the flat, surface of the hemisphere rests on the sphere.

14. A heavy right cone rests with its base on a fixed rough sphere of given radius; find the greatest height of the cone if it be in stable equilibrium.

15. A uniform beam, of thickness $2b$, rests symmetrically on a perfectly rough horizontal cylinder of radius a; shew that the equilibrium of the beam will be stable or unstable according as b is less or greater than a.

16. A heavy uniform cube balances on the highest point of a sphere, whose radius is r. If the sphere be rough enough to prevent sliding, and if the side of the cube be $\frac{\pi r}{2}$, shew that the cube can rock through a right angle without falling.

17. A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere; shew that, if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin\alpha$ be less than $\frac{3r}{a}$, where a is one of the equal sides of the triangle.

18. A weight W is supported on a smooth inclined plane by a given weight P, connected with W by means of a string passing round a fixed pulley whose position is given. Find the position of W on the plane, and determine whether the position is stable or unstable.

19. A rough circular disc, of radius r, is movable about a point distant c from its centre. A string, rough enough to prevent any slipping, hangs over the circumference and carries unequal weights W, w at its ends. Find the position of equilibrium, and determine whether it is stable or unstable.

CHAPTER XI.

MACHINES.

131. In the following chapter we shall explain and discuss the equilibrium of some of the simplest of machines, viz., (1) The Lever, (2) The Pulley and Systems of Pulleys, (3) The Inclined Plane, (4) The Wheel and Axle, (5) The Common Balance, (6) The Steelyards, and (7) The Screw.

We shall suppose the different portions of these machines to be smooth, and that the forces acting on them always balance, so that the machines are at rest.

132. The Lever, The Wheel and Axle, The Balance, and the Steelyards are similar machines. In each we have either a point, or an axis, fixed about which the machine can revolve.

In the pulleys the essential part is a flexible string or strings.

133. When two external forces applied to a machine balance, one is called the Power and the other is called the Weight.

A machine is always used in practice to overcome some resistance; the force we exert on the machine is the power; the resistance to be overcome, in whatever form it may appear, is called the Weight.

16. A heavy uniform cube balances on the high
sphere, whose radius is r. If the sphere be rough eno
sliding, and if the side of the cube be $\frac{\pi r}{2}$, shew that the
through a right angle without falling.

17. A lamina in the form of an isosceles triangle,
angle is α, is placed on a sphere, of radius r, so that its p
and one of its equal sides is in contact with the sphere; sh
triangle be slightly displaced in its own plane, the equili
if $\sin \alpha$ be less than $\frac{3r}{a}$, where a is one of the equal
triangle.

18. A weight W is supported on a smooth inclin
given weight P, connected with W by means of a string
a fixed pulley whose position is given. Find the p
on the plane, and determine whether the position is s
stable.

19. A rough circular disc, of radius r, is movable a
distant c from its centre. A string, rough enough to
slipping, hangs over the circumference and carries unequa
w at its ends. Find the position of equilibrium, and determ
it is stable or unstable.

134. **Mechanical Advantage.** If in any machine a power P balance a weight W, the ratio $W : P$ is called the mechanical advantage of the machine, so that

Weight = Power × Mechanical Advantage.

Almost all machines are constructed so that the mechanical advantage is a ratio greater than unity.

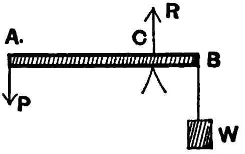
If in any machine the mechanical advantage be less than unity, it may, with more accuracy, be called mechanical disadvantage.

I. The Lever.

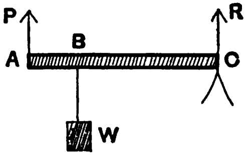
135. The Lever consists essentially of a rigid bar, straight or bent, which has one point fixed about which the rest of the lever can turn. This fixed point is called the Fulcrum, and the perpendicular distances between the fulcrum and the lines of action of the power and the weight are called the arms of the lever.

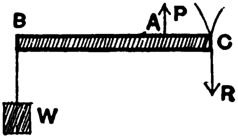
When the lever is straight, and the power and weight act perpendicular to the lever, it is usual to distinguish three classes or orders.

Class I. Here the power P and the weight W act on opposite sides of the fulcrum C.



Class II. Here the power P and the weight W act on the same side of the fulcrum C, but the former acts at a greater distance than the latter from the fulcrum.





Class III. Here the power P and the weight W act on the same side of the fulcrum C, but the former acts at a less distance than the latter from the fulcrum.

136. *Conditions of equilibrium of a straight lever.*

In each case we have three parallel forces acting on the body, so that the reaction, R, at the fulcrum must be equal and opposite to the resultant of P and W.

In the first class P and W are like parallel forces, so that their resultant is $P + W$. Hence

$$R = P + W.$$

In the second class P and W are unlike parallel forces, so that

$$R = W - P.$$

So in the third class $R = P - W$.

In the first and third cases we see that R and P act in opposite directions; in the second class they act in the same direction.

In all three classes, since the resultant of P and W passes through A we have, as in Art. 53,

$$P \, . \, AC = W \, . \, BC,$$

i.e. $P \times$ the arm of $P = W \times$ the arm of W.

Since $\dfrac{W}{P} = \dfrac{\text{arm of } P}{\text{arm of } W}$, we observe that generally in Class I., and always in Class II., there is mechanical advantage, but that in Class III. there is mechanical disadvantage.

The practical use of levers of the latter class is to apply a force at some point at which it is not easy to apply the force directly.

137. In the previous article we have neglected the weight of the bar itself.

If the weight be taken into consideration we must, as in Art. 91, obtain the conditions of equilibrium by equating to zero the algebraic sum of the moments of the forces about the fulcrum F.

138. Examples of the different classes of levers are;

Class I. A Poker (*when used to stir the fire, the bar of the grate being the fulcrum*); A Claw-hammer (*when used to extract nails*); A Crowbar (*when used with a point in it resting on a fixed support*); A Pair of Scales; The Brake of a Pump.

Double examples are; A Pair of Scissors, A Pair of Pincers.

Class II. A Wheelbarrow; A Cork Squeezer; A Crowbar (*with one end in contact with the ground*); An Oar (*assuming the end of the oar in contact with the water to be at rest*).

A Pair of Nutcrackers is a double lever of this class.

Class III. The Treadle of a Lathe; The Human Forearm (*when the latter is used to support a weight placed on the palm of the hand. The Fulcrum is the elbow, and the tension exerted by the muscles is the power*).

A Pair of Sugar-tongs is a double lever of this class.

139. *Bent Levers.*

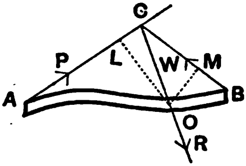
Let AOB be a bent lever, of which O is the fulcrum, and let OL, OM be the perpendiculars from O upon the lines of action AC, BC of the power P and the weight W.

The condition of equilibrium of Art. 91 again applies, and we have, by taking moments about O,

$$P \, . \, OL = W \, . \, OM \, \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)};$$

$$\therefore \frac{P}{W} = \frac{OM}{OL}$$

$$= \frac{\text{perpendicular from fulcrum on direction of the weight}}{\text{perpendicular from fulcrum on direction of the power}}.$$



To obtain the reaction at O let the directions of P and W meet in C. Since there are only three forces acting on the body, the direction of the reaction R at O must pass through C, and then, by Lami's Theorem, we have

$$\frac{R}{\sin ACB} = \frac{P}{\sin BCO} = \frac{W}{\sin ACO}.$$

The reaction may also be obtained, as in Art. 48, by resolving the forces R, P, and W in two directions at right angles.

If the power and weight be parallel forces, the reaction R is then vertical and equal to $(P + Q)$, and, as before, we have

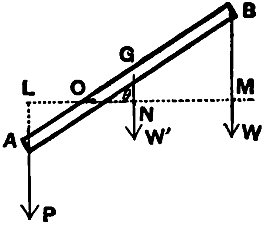
$$P \, . \, OL = W \, . \, OM,$$

where OL and OM are the perpendiculars from O upon the lines of action of the forces.

If the weight W' of the lever be not neglected, we have an additional term to introduce into our equation of moments.

140. *If two weights balance, about a fixed fulcrum, at the extremities of a straight lever, in any position inclined to the vertical, they will balance in any other position.*

Let AB be the lever, of weight W', and let its centre of gravity be G. Let the lever balance about a fulcrum



O in any position inclined at an angle θ to the horizontal, the weights at A and B being P and W respectively.

Through O draw a horizontal line $LONM$ to meet the lines of action of P, W', and W in L, N, and M respectively.

Since the forces balance about O, we have

$$P \,.\, OL = W \,.\, OM + W' \,.\, ON.$$

$$\therefore\; P \,.\, OA \cos\theta = W \,.\, OB \cos\theta + W' \,.\, OG \cos\theta.$$

$$\therefore\; P \,.\, OA = W \,.\, OB + W' \,.\, OG.$$

This condition of equilibrium is independent of the inclination θ of the lever to the horizontal; hence in any other position of the lever the condition would be the same.

Hence, if the lever be in equilibrium in one position, it will be in equilibrium in all positions.

EXAMPLES. XX.

1. In a weightless lever, if one of the forces be equal to 10 lbs. wt. and the pressure on the fulcrum be equal to 16 lbs. wt., and the length of the shorter arm be 3 feet, find the length of the longer arm.

2. Where must the fulcrum be so that a weight of 6 lbs. may balance a weight of 8 lbs. on a straight lever, 7 feet long?

If each weight be increased by 1 lb., in what direction will the lever turn?

3. If two forces, applied to a lever, balance, and if the pressure on the fulcrum be ten times the difference of the forces, find the ratio of the arms.

4. A lever, 1 yard long, has weights of 6 and 20 lbs. fastened to its ends, and balances about a point distant 9 inches from one end; find its weight.

5. A straight lever, *AB*, 12 feet long, balances about a point, 1 foot from *A*, when a weight of 13 lbs. is suspended from *A*. It will balance about a point, which is 1 foot from *B*, when a weight of 11 lbs. is suspended from *B*. Shew that the centre of gravity of the lever is 5 inches from the middle point of the lever.

6. A straight uniform lever is kept in equilibrium by weights of 12 and 5 lbs. respectively attached to the ends of its arms, and the length of one arm is double that of the other. What is the weight of the lever?

7. A straight uniform lever, of length 5 feet and weight 10 lbs., has its fulcrum at one end and weights of 3 and 6 lbs. are fastened to it at distances of 1 and 3 feet respectively from the fulcrum; it is kept horizontal by a force at its other end; find the pressure on the fulcrum.

8. A uniform lever is 18 inches long and is of weight 18 ounces; find the position of the fulcrum when a weight of 27 ounces at one end of the lever balances one of 9 ounces at the other.

If the lesser weight be doubled, by how much must the position of the fulcrum be shifted so as to preserve equilibrium?

9. Two weights, of 8 and 4 ounces, are in equilibrium when attached to the opposite ends of a rod of negligible weight; if 2 ounces be added to the greater, the fulcrum must be moved through $\frac{4}{7}$ths of an inch to preserve equilibrium; find the length of the lever.

10. The short arm of one lever is hinged to the long arm of a second lever, and the short arm of the latter is attached to a press; the long arms being each 3 feet in length, and the short arms 6 inches, find what pressure will be produced on the press by a force, equal to 10 stone weight, applied to the short end of the first lever.

11. A straight heavy uniform lever, 21 inches long, has a fulcrum at its end. A power, equal to the weight of 12 lbs., acting at a distance of 7 inches from the fulcrum, supports a weight of 3 lbs. hanging at the other end of the lever. If the weight be increased by 1 lb., what power at a distance of 5 inches from the fulcrum will support the lever?

12. On a lever, forces of 13 and 14 lbs. weight balance, and they meet at an angle whose cosine is $-\frac{5}{13}$; find the pressure on the fulcrum.

13. A straight lever is acted on, at its extremities, by forces in the ratio $\sqrt{3}+1 : \sqrt{3}-1$, and inclined at angles of 30° and 60° to its length. Find the magnitude of the pressure on the fulcrum, and the direction in which it acts.

14. The arms of a bent lever are at right angles to one another, and the arms are in the ratio of 5 to 1. The longer arm is inclined to the horizon at an angle of 45°, and carries at its end a weight of 10 lbs.; the end of the shorter arm presses against a horizontal plane; find the pressure on the plane.

15. The arms of a uniform heavy bent rod are inclined to one another at an angle of 120°, and their lengths are in the ratio of $2 : 1$; if the rod be suspended from its angular point, find the position in which it will rest.

16. A uniform bar, of length $7\frac{1}{2}$ feet and weight 17 lbs., rests on a horizontal table with one end projecting $2\frac{1}{2}$ feet over the edge; find the greatest weight that can be attached to its end, without making the bar topple over.

17. A straight uniform lever has for its fulcrum a hinge at one end A, and from a point B is hung a body of weight W. If the strain at the hinge must not exceed $\frac{1}{2}W$ in either direction, upwards or downwards, shew that the power must act somewhere within a space equal to $\frac{4}{3}AB$.

18. Shew that the propelling force on an eight-oared boat is 672 lbs. weight, supposing each man to pull his oar with a force of 56 lbs. weight, and that the length of the oar from the middle of the blade to the handle is three times that from the handle to the rowlock.

19. In a pair of nutcrackers, 5 inches long, if the nut be placed at a distance of $\frac{7}{8}$ inch from the hinge, a pressure of $3\frac{1}{2}$ lbs. applied to the ends of the arms will crack the nut. What weight placed on the top of the nut will crack it?

20. A man raises a 3-foot cube of stone, weighing 2 tons, by means of a crowbar, 4 feet long, after having thrust one end of the bar under the stone to a distance of 6 inches; what force must be applied at the other end of the bar to raise the stone?

21. A cubical block, of edge a, is being turned over by a crowbar applied at the middle point of the edge in a plane through its centre of gravity; if the crowbar be held at rest when it is inclined at an angle of 60° to the horizon, the lower face of the block being then inclined at 30° to the horizon, and if the weight of the block be n times the force applied, find the length of the crowbar.

II. Pulleys.

141. A pulley is composed of a wheel of wood, or metal, grooved along its circumference to receive a string or rope; it can turn freely about an axle passing through its centre perpendicular to its plane, the ends of this axle being supported by a frame of wood called the block.

A pulley is said to be movable or fixed according as its block is movable or fixed.

The weight of the pulley is often so small, compared with the weights which it supports, that it may be neglected; such a pulley is called a weightless pulley.

We shall always neglect the weight of the string or rope which passes round the pulley.

We shall also always consider the pulley to be perfectly smooth, so that the tension of a string which passes round a pulley is constant throughout its length.

142. **Single Pulley.** The use of a single pulley is to apply a power in a different direction from that in which it is convenient to us to apply the power.

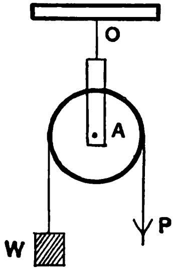
Thus, in the first figure, a man standing on the ground and pulling vertically at one end of the rope might support a weight W hanging at the other end; in the second figure the same man pulling sideways might support the weight.

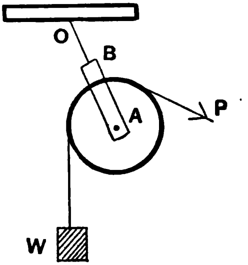
In each case the tension of the string passing round the pulley is unaltered; the power P is therefore equal to the weight W.

In the first figure the action on the fixed support to which the block is attached must balance the other forces on the pulley-block, and must therefore be equal to

$$W + P + w,$$

i.e., $2W + w$, where w is the weight of the pulley-block.





In the second figure, neglecting the weight of the pulley, the power P, and the weight W, being equal, must be equally inclined to the line OA.

Hence, if T be the tension of the supporting string OB and 2θ the angle between the directions of P and W, we have

$$T = P \cos\theta + W \cos\theta = 2W \cos\theta.$$

If w be the weight of the pulley, we should have, similarly,

$$T = 2(W + w)\cos\theta.$$

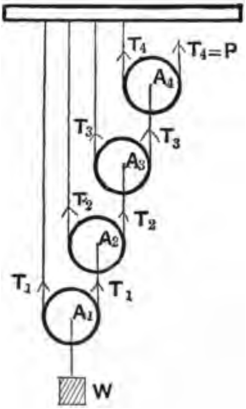
143. We shall discuss three systems of pulleys and shall follow the usual order; there is no particular reason for this order, but it is convenient to retain it for purposes of reference.

144. **First system of Pulleys.** *Each string attached to the supporting beam. To find the relation between the power and the weight.*

In this system of pulleys the weight is attached to the lowest pulley, and the string passing round it has one end attached to the fixed beam, and the other end attached to the next highest pulley; the string passing round the latter pulley has one end attached to the fixed beam, and the other to the next pulley, and so on; the power is applied to the free end of the last string.

Often there is an additional fixed pulley over which the free end of the last string passes; the power may then be applied as a downward force.

Let A_1, A_2, ... be the pulleys, beginning from the lowest, and let the tensions of the strings passing round them be T_1, T_2, ... Let W be the weight and P the power.



[N.B. The string passing round any pulley, A_2 say, pulls A_2 vertically *upwards*, and pulls A_3 *downwards*.]

I. *Let the weights of the pulleys be neglected.*

From the equilibrium of the pulleys A_1, A_2, … , taken in order, we have

$$2T_1 = W; \quad \therefore T_1 = \tfrac{1}{2}W.$$

$$2T_2 = T_1; \quad \therefore T_2 = \tfrac{1}{2}T_1 = \frac{1}{2^2}W.$$

$$2T_3 = T_2; \quad \therefore T_3 = \tfrac{1}{2}T_2 = \frac{1}{2^3}W.$$

$$2T_4 = T_3; \quad \therefore T_4 = \tfrac{1}{2}T_3 = \frac{1}{2^4}W.$$

But, with our figure, $T_4 = P$.

$$\therefore \quad P = \frac{1}{2^4}W.$$

Similarly, if there were n pulleys, we should have

$$P = \frac{1}{2^n}W.$$

Hence, in this system of pulleys, the mechanical advantage

$$= \frac{W}{P} = 2^n.$$

II. *Let the weights of the pulleys in succession, beginning from the lowest, be* w_1, w_2, … .

In this case we have an additional downward force on each pulley.

Resolving as before, we have

$$2T_1 = W + w_1,$$

$$2T_2 = T_1 + w_2,$$

$$2T_3 = T_2 + w_3,$$

$$2T_4 = T_3 + w_4.$$

$$\therefore \quad T_1 = \frac{W}{2} + \frac{w_1}{2},$$

$$T_2 = \tfrac{1}{2}T_1 + \frac{w_2}{2} = \frac{W}{2^2} + \frac{w_1}{2^2} + \frac{w_2}{2},$$

$$T_3 = \tfrac{1}{2}T_2 + \frac{w_3}{2} = \frac{W}{2^3} + \frac{w_1}{2^3} + \frac{w_2}{2^2} + \frac{w_3}{2},$$

and $P = T_4 = \tfrac{1}{2}T_3 + \frac{w_4}{2} = \frac{W}{2^4} + \frac{w_1}{2^4} + \frac{w_2}{2^3} + \frac{w_3}{2^2} + \frac{w_4}{2}.$

Similarly, if there were n pulleys, we should have

$$P = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \ldots + \frac{w_n}{2}.$$

$$\therefore \quad 2^n P = W + w_1 + 2 \,.\, w_2 + 2^2 w_3 + \ldots + 2^{n-1} w_n.$$

If the pulleys be all equal, we have

$$w_1 = w_2 = \ldots = w_n.$$

$$\therefore \quad 2^n P = W + w(1 + 2 + 2^2 + \ldots + 2^{n-1})$$
$$= W + w(2^n - 1),$$

by summing the geometrical progression.

It follows that the mechanical advantage, $\frac{W}{P}$, depends on the weight of the pulleys.

In this system of pulleys we observe that the greater the weight of the pulleys, the greater must P be to support a given weight W; the weights of the pulleys oppose the power, and the pulleys should therefore be made as light as is consistent with the required strength.

Stress on the beam from which the pulleys are hung.

Let R be the stress on the beam. Since R, together with the power P, supports the system of pulleys, together with the weight W, we have

$$R + P = W + w_1 + w_2 + \ldots + w_n.$$

$$\therefore\ R = W + w_1 + w_2 + \ldots + w_n - \frac{W + w_1 + 2w_2 + 2^2 w_3 + \ldots + 2^{n-1} w_n}{2^n}$$

$$= W\left(1 - \frac{1}{2^n}\right) + w_1\left(1 - \frac{1}{2^n}\right) + w_2\left(1 - \frac{1}{2^{n-1}}\right)$$

$$+ w_3\left(1 - \frac{1}{2^{n-2}}\right) + \ldots + w_n\left(1 - \frac{1}{2}\right).$$

Ex. *If there be 4 movable pulleys, whose weights, commencing with the lowest, are 4, 5, 6, and 7 lbs., what power will support a body of weight 1 cwt.?*

Using the notation of the previous article, we have

$$2T_1 = 112 + 4; \qquad \therefore\ T_1 = 58.$$

$$2T_2 = T_1 + 5 = 63; \qquad \therefore\ T_2 = 31\tfrac{1}{2}.$$

$$2T_3 = T_2 + 6 = 37\tfrac{1}{2}; \qquad \therefore\ T_3 = 18\tfrac{3}{4}.$$

$$2P = T_3 + 7 = 25\tfrac{3}{4}; \qquad \therefore\ P = 12\tfrac{7}{8} \text{ lbs. wt.}$$

EXAMPLES. XXI.

1. In the following cases, the movable pulleys are weightless, their number is n, the weight is W, and the power is P;

(1) If $n = 4$; $P = 20$ lbs. wt.; find W;
(2) If $n = 4$; $W = 1$ cwt.; find P;
(3) If $W = 56$ lbs. wt.; $P = 7$ lbs. wt.; find n.

2. In the following cases, the movable pulleys are of equal weight w, and are n in number, P is the power, and W is the weight;

(1) If $n = 4$; $w = 1$ lb. wt.; $W = 97$ lbs. wt.; find P;
(2) If $n = 3$; $w = 1\frac{1}{2}$ lbs. wt.; $P = 7$ lbs. wt.; find W;
(3) If $n = 5$; $W = 775$ lbs. wt.; $P = 31$ lbs. wt.; find w;
(4) If $W = 107$ lbs. wt.; $P = 2$ lbs. wt.; $w = \frac{1}{3}$ lbs. wt.; find n.

3. In the first system of pulleys, if there be 4 pulleys, each of weight 2 lbs., what weight can be raised by a power equal to the weight of 20 lbs.?

4. If there be 3 movable pulleys, whose weights, commencing with the lowest, are 9, 2, and 1 lbs. respectively, what power will support a weight of 69 lbs.?

5. If there be 4 movable pulleys, whose weights, commencing with the lowest, are 4, 3, 2, and 1 lbs. respectively, what power will support a weight of 54 lbs.?

6. If there be 4 movable pulleys, each of weight w, and the power be P, shew that the stress on the beam is $15P - 11w$.

7. If there be 3 movable pulleys and their weights beginning from the lowest be 4, 2, and 1 lbs. respectively, what power will be required to support a weight of 28 lbs.?

8. Shew that, on the supposition that the pulleys are weightless, the mechanical advantage is greater than it actually is.

9. In the system of pulleys in which each hangs by a separate string, if there be 3 pulleys, it is found that a certain weight can be supported by a power of 7 lbs. weight; but, if there be 4 pulleys, the same weight can be supported by a power of 4 lbs. weight; find the weight supported and the weight of the pulleys, which are equal.

10. A system consists of 4 pulleys, arranged so that each hangs by a separate string, one end being fastened to the upper block, and all the free ends being vertical. If the weights of the pulleys, beginning at the lowest, be w, $2w$, $3w$, and $4w$, find the power necessary to support a weight $15w$, and the magnitude of the single force necessary to support the beam to which the other ends of the string are attached.

11. In the system of 4 heavy pulleys, if P be the power and W the weight, shew that the stress on the beam is intermediate between $\frac{15}{16}W$ and $15P$.

12. A man, of 12 stone weight, is suspended from the lowest of a system of 4 weightless pulleys, in which each hangs by a separate string, and supports himself by pulling at the end of the string which passes over the fixed pulley. Find the amount of his pull on this string.

13. A man, whose weight is 156 lbs., is suspended from the lowest of a system of 4 pulleys, each being of weight 10 lbs., and supports himself by pulling at the end of the string which passes over the fixed pulley. Find the force which he exerts on the string, supposing all the strings to be vertical.

145. **Second system of pulleys.** *The same string passing round all the pulleys. To find the relation between the power and the weight.*

In this system there are two blocks, each containing pulleys, the upper block being fixed and the lower block movable. The same string passes round all the pulleys as in the figures.

If the number of pulleys in the upper block be the same as in the lower block (Fig. 1), one end of the string must be fastened to the upper block; if the number in the upper block be greater by one than the number in the lower block (Fig. 2), the end of the string must be attached to the lower block.

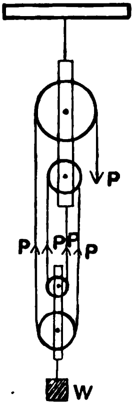


Fig. 1.

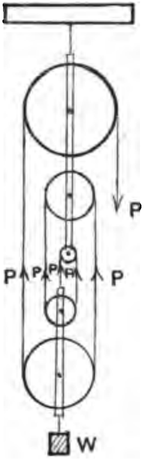


Fig. 2.

In the first case, the number of portions of string connecting the blocks is even; in the second case, the number is odd.

In either case, let n be the number of portions of string at the lower block. Since we have only one string passing over smooth pulleys, the tension of each of these portions is P, so that the total upward force at the lower block is $n \, . \, P$.

Let W be the weight supported, and w the weight of the lower block.

Hence $W + w = nP$, giving the relation required.

In practice the pulleys of each block are often placed parallel to one another, so that the strings are not mathematically parallel; they are, however, very approximately parallel, so that the above relation is still very approximately true.

EXAMPLES. XXII.

1. If a weight of 5 lbs. support a weight of 24 lbs., find the weight of the lower block, when there are 3 pulleys in each block.

✓ 2. If weights of 5 and 6 lbs. respectively at the free ends of the string support weights of 18 and 22 lbs. at the lower block, find the number of the strings and the weight of the lower block.

3. If weights of 4 lbs. and 5 lbs. support weights of 5 lbs. and 18 lbs. respectively, what is the weight of the lower block, and how many pulleys are there in it?

✓ 4. A power of 6 lbs. just supports a weight of 28 lbs., and a power of 8 lbs. just supports a weight of 42 lbs.; find the number of strings and the weight of the lower block.

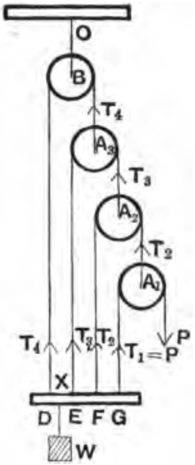
5. In the second system of pulleys, if a basket be suspended from the lower block and a man in the basket support himself and the basket, by pulling at the free end of the string, find the tension he exerts, neglecting the inclination of the string to the vertical, and assuming the weight of the man and basket to be W.

6. A man, whose weight is 12 stone, raises 3 cwt. by means of a system of pulleys in which the same string passes round all the pulleys, there being 4 in each block, and the string being attached to the upper block; neglecting the weights of the pulleys, find what will be his pressure on the ground if he pull vertically downwards.

7. We are told that the cable by which "Great Paul," whose weight is 18 tons, was lifted into its place in the cathedral tower, passed four times through the two blocks of pulleys. From this statement give a description of the pulleys, and estimate the strength of the cable.

146. **Third system of pulleys.** *All the strings attached to the weight. To find the relation between the power and the weight.*

In this system the string passing round any pulley is attached at one end to a bar, from which the weight is suspended, and at the other end to the next lower pulley; the string round the lowest pulley is attached at one end to the bar, whilst at the other end of this string the power is applied.



In this system the upper pulley is a fixed pulley.

Let A_1, A_2, A_3 ... be the movable pulleys, beginning from the lowest, and let the tensions of the strings passing round these pulleys respectively be

$$T_1, \quad T_2, \quad T_3 \dots .$$

If the power be P, we have clearly

$$T_1 = P.$$

I. *Let the weights of the pulleys be neglected.*

For the equilibrium of the pulleys, taken in order and commencing from the lowest, we have

$$T_2 = 2T_1 = 2\,P,$$
$$T_3 = 2T_2 = 2^2P,$$
and
$$T_4 = 2T_3 = 2^3P.$$

But, since the bar, from which W is suspended, is in equilibrium, we have

$$\begin{aligned} W &= T_1 + T_2 + T_3 + T_4 \\ &= P + 2P + 2^2P + 2^3P \\ &= P\frac{2^4-1}{2-1} = P\,(2^4-1) \text{ ....} (1). \end{aligned}$$

If there were n pulleys, of which $(n-1)$ would be movable, we should have, similarly,

$$\begin{aligned} W &= T_1 + T_2 + T_3 + \ldots + T_n \\ &= P + 2P + 2^2P + \ldots + 2^{n-1}P \\ &= P\left[\frac{2^n-1}{2-1}\right], \end{aligned}$$

by summing the geometrical progression,

$$= P\,(2^n - 1) \text{ ....} (2).$$

Hence the mechanical advantage is $2^n - 1$.

II. *Let the weights of the movable pulleys, taken in order and commencing with the lowest, be* w_1, w_2, ...

Considering the equilibrium of the pulleys in order, we have

$$T_2 = 2T_1 + w_1 = 2P + w_1,$$
$$T_3 = 2T_2 + w_2 = 2^2P + 2w_1 + w_2,$$
$$T_4 = 2T_3 + w_3 = 2^3P + 2^2w_1 + 2w_2 + w_3.$$

But, from the equilibrium of the bar,

$$\begin{aligned}W &= T_4 + T_3 + T_2 + T_1\\ &= (2^3 + 2^2 + 2 + 1)P + (2^2 + 2 + 1)w_1 + (2+1)w_2 + w_3\\ &= \frac{2^4-1}{2-1}P + \frac{2^3-1}{2-1}w_1 + \frac{2^2-1}{2-1}w_2 + w_3\\ &= (2^4-1)P + (2^3-1)w_1 + (2^2-1)w_2 + w_3 \ \ldots\ldots\ldots(3).\end{aligned}$$

If there were n pulleys, of which $(n-1)$ would be movable, we should have, similarly,

$$\begin{aligned}W &= T_n + T_{n-1} + \ldots + T_2 + T_1\\ &= (2^{n-1} + 2^{n-2} + \ldots + 1)P + (2^{n-2} + 2^{n-3} + \ldots + 1)w_1\\ &\qquad + (2^{n-3} + 2^{n-4} + \ldots + 1)w_2 + \ldots + (2+1)w_{n-2} + w_{n-1}\\ &= \frac{2^n-1}{2-1}P + \frac{2^{n-1}-1}{2-1}w_1 + \frac{2^{n-2}-1}{2-1}w_2\\ &\qquad + \ldots + \frac{2^2-1}{2-1}w_{n-2} + w_{n-1}\\ &= (2^n-1)P + (2^{n-1}-1)w_1 + (2^{n-2}-1)w_2 + \ldots\\ &\qquad + (2^2-1)w_{n-2} + (2-1)w_{n-1} \ldots\ldots(4).\end{aligned}$$

If the pulleys be all equal, so that

$$w_1 = w_2 = \ldots = w_{n-1} = w,$$

the relation (4) becomes

$$\begin{aligned}W &= (2^n-1)P + w\,[2^{n-1} + 2^{n-2} + \ldots + 2 - (n-1)]\\ &= (2^n-1)P + w\,[2^n - n - 1],\end{aligned}$$

by summing the geometrical progression.

Stress on the supporting beam. This stress balances the power, the weight, and the weight of the pulleys, and therefore equals

$$P + W + w_1 + w_2 + \ldots + w_n,$$

and hence is easily found.

Ex. *If there be 4 pulleys, whose weights, commencing with the lowest, are 4, 5, 6, and 7 lbs., what power will support a body of weight 1 cwt.?*

Using the notation of the previous article, we have

$$T_2 = 2P + 4,$$

$$T_3 = 2T_2 + 5 = 4P + 13,$$

$$T_4 = 2T_3 + 6 = 8P + 32.$$

Also $$112 = T_4 + T_3 + T_2 + P = 15P + 49.$$

$$\therefore\ P = \frac{63}{15} = 4\tfrac{1}{5} \text{ lbs. wt.}$$

147. In this system we observe that, the greater the weight of each pulley, the less is P required to be in order that it may support a given weight W. Hence the weights of the pulleys assist the power.

If the weights of the pulleys be properly chosen, the system will remain in equilibrium without the application of any power whatever.

For example, suppose we have 3 movable pulleys, each of weight w, the relation (3) of the last Article will become

$$W = 15P + 11w.$$

Hence, if $11w = W$, we have P zero, *i.e.*, if the weight to be supported be eleven times the weight of each of the three movable pulleys, no power need be applied at the free end of the string to preserve equilibrium.

148. In the third system of pulleys, the bar supporting the weight W will not remain horizontal, unless the point at which the weight is attached be properly chosen.

In any particular case the proper point of attachment can be easily found.

Taking the figure of Art. 146 let there be three movable pulleys, whose weights are negligible. Let the distances between the points D, E, F, G at which the strings are attached, be successively a, and let the point at which the weight is attached be X.

The resultant of T_1, T_2, T_3 and T_4 must pass through X.

$\therefore$ by Art. 109,

$$DX = \frac{T_4 \times 0 + T_3 \times a + T_2 \times 2a + T_1 \times 3a}{T_4 + T_3 + T_2 + T_1}$$

$$= \frac{4P \,.\, a + 2P \,.\, 2a + P \,.\, 3a}{8P + 4P + 2P + P} = \frac{11a}{15}.$$

$\therefore$ $DX = \frac{11}{15} DE$, giving the position of X.

EXAMPLES. XXIII.

1. In the following cases, the pulleys are weightless and n in number, P is the power, and W the weight;

(1) If $n = 4$; $P = 2$ lbs. wt.; find W;
(2) If $n = 5$; $W = 124$ lbs. wt.; find P;
(3) If $W = 105$ lbs.; $P = 7$ lbs. wt.; find n.

2. In the following cases, the pulleys are equal and each of weight w, P is the power, and W is the weight;

(1) If $n = 4$; $w = 1$ lb. wt.; $P = 10$ lbs. wt.; find W;
(2) If $n = 3$; $w = \frac{1}{2}$ lb. wt.; $W = 114$ lbs. wt.; find P;
(3) If $n = 5$; $P = 3$ lbs. wt.; $W = 106$ lbs. wt.; find w;
(4) If $P = 4$ lbs. wt.; $W = 137$ lbs. wt.; $w = \frac{1}{2}$ lb. wt.; find n.

3. If there be 5 pulleys, each of weight 1 lb., what power is required to support 3 cwt.?

If the pulleys be of equal size, find to what point of the bar the weight must be attached, so that the beam may be always horizontal.

4. If the strings passing round a system of 4 pulleys be fastened to a rod without weight at distances successively an inch apart, find to what point of the rod the weight must be attached, so that the rod may be always horizontal.

5. Find the mechanical advantage, when the pulleys are 4 in number, and each is of weight $\frac{1}{64}$th that of the weight.

6. In a system of 3 pulleys, in which each string is attached to a bar which carries the weight, if the diameter of each pulley be 2 inches, find to what point of the bar the weight should be attached so that the bar may be always horizontal.

7. If the pulleys be equal, and the power be equal to the weight of one of them, and the number of pulleys be 5, shew that the weight is 57 times the power.

8. In the third system of 3 pulleys, if the weights of the pulleys be all equal, find the relation of the power to the weight when equilibrium is established. If each pulley weigh 2 ounces, what weight would be supported by the pulleys only?

If the weight supported be 25 lbs. wt., and the power be 3 lbs. wt., find what must be the weight of each pulley.

9. In the third system of pulleys, the weight is supported by a power of 70 lbs. The hook by which one of the strings is attached to the weight breaks, and the string is then attached to the pulley which it passed over, and a power of 150 lbs. is now required. Find the number of pulleys and the weight supported.

10. In the third system of pulleys, if the string round the last pulley be tied to the weight, shew that the tension of the string is diminished in a ratio depending on the number of pulleys.

If the tension be decreased in the ratio 16 : 15, find the number of pulleys.

11. In the system of pulleys in which each pulley is attached to the weight, if each pulley have a weight w, and the sum of the weights of the pulleys be W', and P and W be the power and weight in this case, shew that the power $P+w$ would support the weight $W+W'$ in the same system if the pulleys had no weight.

12. If there be n weightless pulleys and if a string, whose ends are attached to the weights P and W, carry a pulley from which a weight W' is suspended, find the relation between P, W, and W'.

13. If there be n pulleys, each of diameter $2a$ and of negligible weight, shew that the distance of the point of application of the weight from the line of action of the power should be $\frac{2^n}{2^n-1}na$.

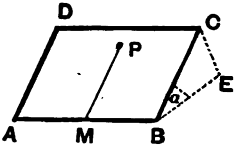
III. The Inclined Plane.

149. The Inclined Plane, considered as a mechanical power, is a rigid plane inclined at an angle to the horizon.

It is used to facilitate the raising of heavy bodies.

In the present chapter we shall only consider the case of a body resting on the plane, and acted upon by forces in a plane perpendicular to the intersection of the inclined plane and the horizontal, *i.e.*, in a vertical plane through the line of greatest slope.

The reader can picture to himself the line of greatest slope on an inclined plane in the following manner: take a rectangular sheet of cardboard $ABCD$, and place it at an angle to the horizontal, so that the line AB is in contact with a horizontal table: take any point P on the cardboard and draw PM perpendicular to the line AB; PM is the line of greatest slope passing through the point P.



From C draw CE perpendicular to the horizontal plane through AB, and join BE. The lines BC, BE, and CE are called respectively the length, base, and height of the inclined plane; also the angle CBE is the inclination of the plane to the horizon.

150. The inclined plane is supposed to be smooth, so that the only reaction between it and any body resting on it is perpendicular to the inclined plane.

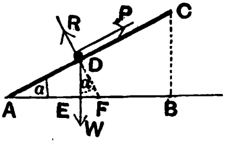
Since the plane is rigid, it is capable of exerting any reaction, however great, that may be necessary to give equilibrium.

151. *A body, of given weight, rests on an inclined plane; to determine the relations between the power, the weight, and the reaction of the plane.*

Let W be the weight of the body, P the power, and R the reaction of the plane; also let α be the inclination of the plane to the horizon.

Case I. *Let the power act up the plane along the line of greatest slope.*

Let AC be the inclined plane, AB the horizontal line



through A, DE a vertical line, and let the perpendicular to the plane through D meet AB in F.

Then clearly

$$\angle FDE = \frac{\pi}{2} - \angle ADE = \angle DAE = \alpha.$$

By Lami's Theorem, (Art. 40), since only three forces act on the body, each is proportional to the sine of the angle between the other two.

$$\therefore \frac{P}{\sin(R, W)} = \frac{R}{\sin(W, P)} = \frac{W}{\sin(P, R)},$$

$$i.e., \frac{P}{\sin(\pi - \alpha)} = \frac{R}{\sin\left(\frac{\pi}{2} + \alpha\right)} = \frac{W}{\sin\frac{\pi}{2}},$$

$$i.e., \frac{P}{\sin\alpha} = \frac{R}{\cos\alpha} = W \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\therefore P = W\sin\alpha, \text{ and } R = W\cos\alpha.$$

The relation (1) may be written in the form

$$P : R : W$$

:: Height of plane : Base of plane : Length of plane.

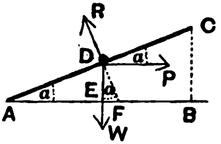
Otherwise thus: Resolve W along and perpendicular to the plane; its components are

$W\cos ADE$, *i.e.*, $W\sin\alpha$, along DA,

and $W\sin ADE$, *i.e.*, $W\cos\alpha$, along DF.

Hence $P = W\sin\alpha$, and $R = W\cos\alpha$.

Case II. *Let the power act horizontally.*



[In this case we must imagine a small hole in the plane at D through which a string is passed and attached to the body, or else that the body is *pushed* toward the plane by a horizontal force.]

As in Case I., we have

$$\frac{P}{\sin(R,\ W)}=\frac{R}{\sin(W,\ P)}=\frac{W}{\sin(P,\ R)},$$

$$i.e.,\quad \frac{P}{\sin(\pi-a)}=\frac{R}{\sin\frac{\pi}{2}}=\frac{W}{\sin\left(\frac{\pi}{2}+a\right)},$$

$$i.e.,\quad \frac{P}{\sin a}=\frac{R}{1}=\frac{W}{\cos a}\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\therefore\ P=W\tan a,\quad\text{and}\quad R=W\sec a.$$

The relation (1) may be written in the form

$$P : R : W$$

:: Height of Plane : Length of Plane : Base of Plane.

Otherwise thus: The components of W along and perpendicular to the plane are $W\sin a$ and $W\cos a$; the components of P, similarly, are $P\cos a$ and $P\sin a$.

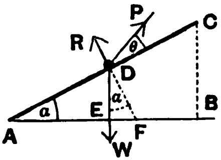
$$\therefore\ P\cos a=W\sin a,$$

and

$$R=P\sin a+W\cos a=W\left[\frac{\sin^2 a}{\cos a}+\cos a\right]=W\frac{\sin^2 a+\cos^2 a}{\cos a}=W\sec a.$$

$$\therefore\ P=W\tan a,\quad\text{and}\quad R=W\sec a.$$

Case III. *Let the power act at an angle θ with the inclined plane.*



By Lami's Theorem we have

$$\frac{P}{\sin(R,\ W)}=\frac{R}{\sin(W,\ P)}=\frac{W}{\sin(P,\ R)},$$

$$i.e.,\quad \frac{P}{\sin(\pi-\alpha)}=\frac{R}{\sin\left(\frac{\pi}{2}+\theta+\alpha\right)}=\frac{W}{\sin\left(\frac{\pi}{2}-\theta\right)},$$

$$i.e.,\quad \frac{P}{\sin\alpha}=\frac{R}{\cos(\theta+\alpha)}=\frac{W}{\cos\theta}.$$

$$\therefore\ P=W\frac{\sin\alpha}{\cos\theta},\ \text{and}\ R=W\frac{\cos(\theta+\alpha)}{\cos\theta}.$$

Otherwise thus: Resolving along and perpendicular to the plane, we have

$$P\cos\theta=W\sin\alpha,$$

and

$$R+P\sin\theta=W\cos\alpha.$$

$$\therefore\ P=W\frac{\sin\alpha}{\cos\theta},$$

and

$$R=W\cos\alpha-P\sin\theta=W\left[\cos\alpha-\frac{\sin\alpha\sin\theta}{\cos\theta}\right]$$

$$=W\frac{\cos\alpha\cos\theta-\sin\alpha\sin\theta}{\cos\theta}=W\frac{\cos(\alpha+\theta)}{\cos\theta}.$$

It will be noted that Case III. includes both Cases I. and II.; if we make θ zero, we obtain Case I.; if we put θ equal to $(-\alpha)$, we have Case II.

152. If the power does not act in a vertical plane through the line of greatest slope there could not be equilibrium on a smooth inclined plane; in this case we could, however, have equilibrium with a rough inclined plane. We shall return to this case in the next chapter.

EXAMPLES. XXIV.

1. What force, acting horizontally, could keep a mass of 16 lbs. at rest on a smooth inclined plane, whose height is 3 feet and length of base 4 feet, and what is the pressure on the plane?

2. A body rests on an inclined plane, being supported by a force acting up the plane equal to half its weight. Find the inclination of the plane to the horizon and the reaction of the plane.

3. A rope, whose inclination to the vertical is 30°, is just strong enough to support a weight of 180 lbs. on a smooth plane, whose inclination to the horizon is 30°. Find approximately the greatest tension that the rope could exert.

4. A body rests on a plane, inclined at an angle of 60° to the horizon, and is supported by a force inclined at an angle of 30° to the horizon; shew that the force and the reaction of the plane are each equal to the weight of the body.

5. A body, of weight $2P$, is kept in equilibrium on an inclined plane by a horizontal force P, together with a force P acting parallel to the plane; find the ratio of the base of the plane to the height and also the pressure on the plane.

6. A body rests on a plane, inclined to the horizon at an angle of 30°, being supported by a power inclined at 30° to the plane; find the ratio of the weight of the body to the power.

7. A weight is supported on an inclined plane by a force inclined to the plane; if the weight, the force, and the pressure be as the numbers 4, 3, and 2, find the inclination of the plane and the direction of the force.

8. A body, of 5 lbs. wt., is placed on a smooth plane inclined at 30° to the horizon, and is acted on by two forces, one equal to the weight of 2 lbs. and acting parallel to the plane and upwards, and the other equal to P and acting at an angle of 30° with the plane. Find P and the pressure on the plane.

9. Find the force which acting up an inclined plane will keep a body, of 10 lbs. weight, in equilibrium, it being given that the force, the pressure on the plane, and the weight of the body are in arithmetical progression.

10. If a power P, acting parallel to an inclined plane and supporting a mass of weight W, produces on the plane a pressure R, shew that the same power, acting horizontally and supporting a mass of weight R, will produce on the plane a pressure W.

11. Two boards, of lengths 11 and 8 feet, are fixed with their lower ends on a horizontal plane and their upper ends in contact; on these planes rest bodies of weights W and 12 lbs. respectively, which are connected by a string passing over the common vertex of the boards; find the value of W.

12. A number of loaded trucks, each containing 1 ton, on one part of a tramway inclined at an angle α to the horizon supports an equal number of empty trucks on another part whose inclination is β. Find the weight of a truck.

13. A body rests on a plane inclined to the horizon at an angle α; if the pressure on the plane be equal to the power applied, shew that the inclination of the power to the inclined plane is $\frac{\pi}{2} - 2\alpha$.

14. A heavy string is placed with a portion of it resting on a given inclined plane, the remaining part hanging vertically over a small pulley at the top of the plane. Find what point of the string should be placed over the pulley for equilibrium.

15. On two inclined planes, of equal height, two weights are respectively supported, by means of a string passing over the common vertex and parallel to the planes; the length of one plane is double its height, and the length of the other plane is double its base; shew that the pressure on one plane is three times the pressure on the other.

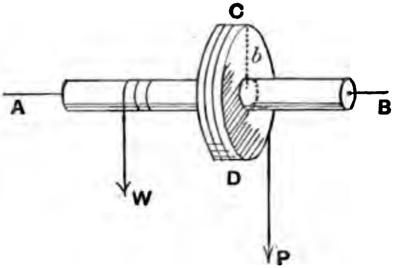
16. A body, of weight 50 lbs., is in equilibrium on a smooth plane inclined at an angle of 20° 20′ to the horizon being supported by a force acting up the plane; find the force and the pressure on the plane.

17. A body, of weight 20 lbs., rests on a smooth plane inclined at an angle of 25° to the horizon, being supported by a force P acting at an angle of 35° with the plane; find P and the pressure on the plane.

18. A body, of weight 30 lbs., rests on a smooth plane inclined at an angle of 28° 15′ to the horizon, being supported by a horizontal force P; find P and the pressure on the plane.

IV. The Wheel and Axle.

153. This machine consists of a strong circular cylinder, or axle, terminating in two pivots, *A* and *B*, which can turn freely on fixed supports. To the cylinder is rigidly attached a wheel, *CD*, the plane of the wheel being perpendicular to the axle.



Round the axle is coiled a rope, one end of which is firmly attached to the axle, and the other end of which is attached to the weight.

Round the circumference of the wheel, in a direction opposite to that of the first rope, is coiled a second rope, having one end firmly attached to the wheel, and having the power applied at its other end. The circumference of the wheel is grooved to prevent the rope from slipping off.

154. *To find the relation between the power and the weight.*

In Art. 93, we have shewn that a body, which can turn freely about a fixed axis, is in equilibrium if the algebraic sum of the moments of the forces about the axis vanishes. In this case, the only forces acting on the machine are the power P and the weight W, which tend to turn the machine in opposite directions. Hence, if a be the radius of the axle, and b be the radius of the wheel, the condition of equilibrium is

$$P \, . \, b = W \, . \, a.$$

Hence the mechanical advantage $= \frac{W}{P}$

$$= \frac{b}{a} = \frac{\text{radius of the wheel}}{\text{radius of the axle}}.$$

155. Theoretically, by making the quantity $\frac{b}{a}$ very large, we can make the mechanical advantage as great as we please; practically however there are limits. Since the pressure of the fixed supports on the axle must balance P and W, it follows that the thickness of the axle, *i.e.*, $2a$, must not be reduced unduly, for then the axle would break. Neither can the radius of the wheel in practice become very large, for then the machine would be unwieldy. Hence the possible values of the mechanical advantage are bounded, in one direction by the strength of our materials, and in the other direction by the necessity of keeping the size of the machine within reasonable limits.

156. In Art. 154 we have neglected the thicknesses of the ropes. If, however, they are too great to be neglected, compared with the radii of the wheel and axle, we may take them into consideration by supposing the tensions of the ropes to act along their middle threads.

Suppose the radii of the ropes which pass round the axle and wheel to be x and y respectively; the distances from the line joining the pivots at which the tensions now act are $(a + x)$ and $(b + y)$ respectively. Hence the condition of equilibrium is

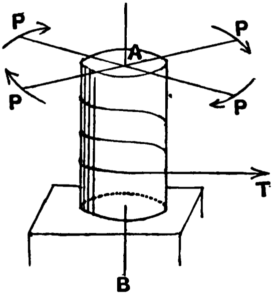
$$P(b + y) = W(a + x),$$

so that

$$\frac{P}{W} = \frac{\text{sum of the radii of the axle and its rope}}{\text{sum of the radii of the wheel and its rope}}.$$

157. Other forms of the Wheel and Axle are the Windlass and Capstan. In these machines the power instead of being applied, as in Art. 153, by means of a rope passing round a cylinder, is applied at the ends of a spoke, or spokes, which are inserted in a plane perpendicular to the axle.

In the Windlass the axle is horizontal, and in the Capstan it is vertical.

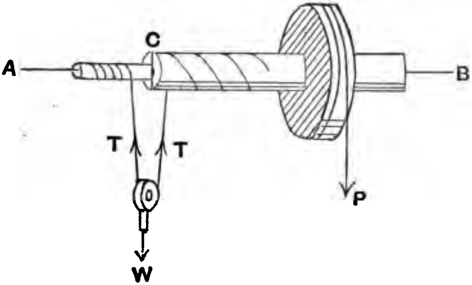


In the latter case the "weight" consists of the tension T of the rope round the axle, and the power consists of the forces applied at the ends of bars inserted into sockets at the point A of the axle. The condition of equilibrium may be obtained as in Art. 154.

*158. *Differential Wheel and Axle.* A slightly modified form of the ordinary wheel and axle is the differential wheel and axle. In this machine the axle consists of two cylinders, having a common axis, joined at their ends, the radii of the two cylinders being different. One end of the rope is wound round one of these cylinders, and its other end is wound in a contrary direction round the other cylinder. Upon the slack portion of the rope is slung a pulley to which the weight is attached. The part of the rope which passes round the smaller cylinder tends to turn the machine in the same direction as the power.

As before, let b be the radius of the wheel and let a and c be the radii of the portion AC and CB of the axle, a being the smaller.

Since the pulley is smooth, the tension T of the string round it is the same throughout its length, and hence, for the equilibrium of the weight, we have $T = \frac{1}{2}W$.



Taking moments about the line AB for the equilibrium of the machine, we have

$$P . b + T . a = T . c.$$

$$\therefore\ P = T\frac{c-a}{b} = \frac{W}{2}\frac{c-a}{b}.$$

Hence the mechanical advantage

$$= \frac{W}{P} = \frac{2b}{c-a}.$$

By making the radii c and a of the two portions of the axle very nearly equal, we can make the mechanical advantage very great, without unduly weakening the machine.

EXAMPLES. XXV.

1. If the radii of the wheel and axle be respectively 2 feet and 3 inches, find what power must be applied to raise a weight of 56 lbs.

2. If the radii of the wheel and axle be respectively 30 inches and 5 inches, find what weight would be supported by a force equal to the weight of 20 lbs., and find also the pressures on the supports on which the axle rests.

If the thickness of the ropes be each 1 inch, find what weight would now be supported.

3. If by means of a wheel and axle a power equal to 3 lbs. weight balance a weight of 30 lbs., and if the radius of the axle be 2 inches, what is the radius of the wheel?

4. The axis of a capstan is 16 inches in diameter and there are 8 bars. At what distance from the axle must 8 men push, 1 at each bar and each exerting a force equal to the weight of 26⅔ lbs., in order that they may just produce a strain sufficient to raise the weight of 1 ton?

5. Four sailors raise an anchor by means of a capstan, the radius of which is 4 ins. and the length of the spokes 6 feet; if each man exert a force equal to the weight of 112 lbs., find the weight of the anchor.

6. Four wheels and axles, in each of which the radii are in the ratio of 5 : 1, are arranged so that the circumference of each axle is applied to the circumference of the next wheel; what power is required to support a weight of 1875 lbs.?

7. The radii of a wheel and axle are 2 feet and 2 ins. respectively, and the strings which hang from them are tied to the two ends of a uniform rod, 2 feet 2 ins. in length and 10 lbs. in weight; what weight must be also hung from one of the strings that the rod may hang in a horizontal position?

8. A pulley is suspended by a vertical loop of string from a wheel-and-axle and supports a weight of 1 cwt., one end of the string being wound round the axle and the other in a contrary direction round the wheel. Find the power which acting at one end of an arm, 2 feet in length, so as to turn the axle, will support the weight, assuming the radii of the wheel and axle to be 1 foot and 2 ins.

9. In the Differential Wheel and Axle, if the radius of the wheel be 1 foot and the radii of the two portions of the axle be 5 and 4 ins. respectively, what power will support a weight of 56 lbs.?

10. In the Differential Wheel and Axle, if the radius of the wheel be 18 ins. and the radii of the two portions of the axle be 6 and 4 ins. respectively, what weight will be supported by a power equal to 20 lbs. weight?

11. In a wheel and axle the radius of the wheel is 1 foot and that of the axle is 1 inch; if 2 weights, each 10 lbs., be fastened to 2 points on the rim of the wheel, so that the line joining them subtends an angle of 120° at the centre of the wheel, find the greatest weight which can be supported by a string hanging from the axle in the usual way.

V. The Common Balance.

159. The Common Balance consists of a rigid beam AB (Art. 160), carrying a scale-pan suspended from each end, which can turn freely about a fulcrum O outside the beam. The fulcrum and the beam are rigidly connected and, if the balance be well constructed, at the point O is a hard steel wedge, whose edge is turned downward and which rests on a small plate of agate.

The body to be weighed is placed in one scale-pan and in the other are placed weights, whose magnitudes are known; these weights are adjusted until the beam of the balance rests in a horizontal position. If OH be perpendicular to the beam, and the arms HA, HB be of equal length, and if the centre of gravity of the beam lie in the line OH, and the scale-pans be of equal weight, then the weight of the body is the same as the sum of the weights placed in the other scale-pan.

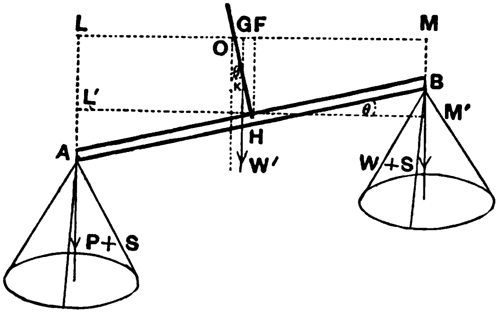
If the weight of the body be not equal to the sum of the weights placed in the other scale-pan, the balance will not rest with its beam horizontal, but will rest with the beam inclined to the horizon.

In the best balances the beam is usually provided with a long pointer attached to the beam at H. The end of this pointer travels along a graduated scale and, when the beam is horizontal, the pointer is vertical and points to the zero graduation on the scale.

160. *To find the position of equilibrium of a balance when the weights placed in the scale-pans are not equal.*

Let the weights placed in the scale-pans be P and W, the former being the greater; let S be the weight of each scale-pan, and let the weight of the beam (and the parts rigidly connected with it) be W', acting at a point K on OH.

When in equilibrium let the beam be inclined at an angle θ to the horizontal, so that OH is inclined at the same angle θ to the vertical.



Let OH and OK be h and k respectively, and let the length of AH or HB be a.

Let horizontal lines through O and H meet the vertical lines through the ends A and B of the beam in the points L, M, L', M' respectively.

Also let the vertical lines through H and K meet LM in F and G respectively.

When the system is in equilibrium, the moments of the forces about O must balance.

$$\therefore\ (P+S)\,.\,OL = (W+S)\,OM + W'\,.\,OG,$$

$$i.e.,\ (P+S)(FL-FO) = (W+S)(FM+OF) + W'\,.\,OG,$$

$$\therefore\ (P+S)(a\cos\theta - h\sin\theta) = (W+S)(a\cos\theta + h\sin\theta) + W'\,.\,k\sin\theta.$$

[For $OF = OH \cos FOH = h \sin\theta$; $OG = OK \sin\theta$;

and $$FL = HL' = a\cos\theta.]$$

$$\therefore\ a\cos\theta\,(P - W) = \sin\theta\,[W'k + (P + W + 2S)\,h].$$

$$\therefore\ \tan\theta = \frac{(P - W)\,a}{W'k + (P + W + 2S)\,h}.$$

161. *Requisites of a good balance.*

(1) The balance must be **true.**

This will be the case if the arms of the balance be equal, if the weights of the scale-pans be equal, and if the centre of gravity of the beam be on the line through the fulcrum perpendicular to the beam; for the beam will now be horizontal when equal weights are placed in the scale-pans.

To test whether the balance is true, first make the beam horizontal by putting sufficient weights in one scale-pan to balance the weight of a body placed in the other; now interchange the body and the weights; if they still balance one another, the balance must be true; if in the second case the beam assumes any position inclined to the vertical, the balance is not true.

(2) The balance must be **sensible,** *i.e.*, the beam must, for any difference, however small, between the weights in the scale-pans, be inclined at an appreciable angle to the horizon.

For a *given* difference between P and W, the greater the inclination of the beam to the horizon the more sensible is the balance; also the less the difference between the weights required to produce a given inclination θ, the greater is the sensibility of the balance.

Hence, when $P-W$ is given, the sensibility varies as $\tan\theta$; also when θ is given it varies as

$$\frac{1}{P-W}.$$

The sensibility is therefore appropriately measured by

$$\frac{\tan\theta}{P-W},$$

i.e. by $$\frac{a}{W'k+(P+W+2S)h}. \quad \text{(Art. 160.)}$$

Hence, to make the balance as sensible as possible, the arm a should be fairly long and the distances h and k should be small, and the weight W' of the beam as small as is consistent with the length and rigidity of the machine.

We must not make h and k zero; for then the points O and K would both coincide with H. In this case the balance would either when the weights in the scale-pans were equal, be, as in Art. 140, in equilibrium in *any* position or else, if the weights in the scale-pans were not equal, it would take up a position as nearly vertical as the mechanism of the machine would allow.

(3) The balance must be **stable** and must quickly take up its position of equilibrium.

The determination of the time taken by the machine to take up its position of equilibrium is essentially a dynamical question. We may however assume that this condition is best satisfied when the moment of the forces about the fulcrum O is greatest. When the weights in the scale-pans are each P, the moment of the forces tending to restore equilibrium

$$= (P+S)(a\cos\theta + h\sin\theta) - (P+S)(a\cos\theta - h\sin\theta) + W'.k\sin\theta$$

$$= [2(P+S)h + W'.k]\sin\theta.$$

This expression is greatest when h and k are greatest.

Since the balance is most sensible when h and k are small, and most stable when these quantities are large, we see that in any balance great sensitiveness and quick weighing are to a certain extent incompatible. In practice this is not very important; for in balances where great sensitiveness is required (such as balances used in a laboratory) we can afford to sacrifice quickness of weighing; the opposite is the case when the balance is used for ordinary commercial purposes.

To insure as much as possible both the qualities of sensibility and quick weighing, the balance should be made with fairly light long arms, and at the same time the distance of the fulcrum from the beam should be considerable.

162. **Double weighing.** By this method the weight of a body may be accurately determined even if the balance be not accurate.

Place the body to be weighed in one scale-pan and in the other pan put sand, or other suitable material, sufficient to balance the body. Next remove the body, and in its place put known weights sufficient to again balance the sand. The weight of the body is now clearly equal to the sum of the weights.

This method is used even in the case of extremely good machines when very great accuracy is desired.

163. **Ex. 1.** *The arms of a balance are equal in length but the beam is unjustly loaded; if a body be placed in each scale-pan in succession and weighed, shew that its true weight is the arithmetic mean between its apparent weights.*

For let the length of the arms be a, and let the distance of the centre of gravity of the beam from the fulcrum be x.

Let a body, whose true weight is W, appear to weigh W_1 and W_2 successively.

If W' be the weight of the beam, we have

$$W\,.\,a = W'\,.\,x + W_1\,.\,a,$$

and

$$W_2\,.\,a = W'\,.\,x + W\,.\,a.$$

Hence, by subtraction,

$$(W - W_2)\,a = (W_1 - W)\,a.$$

$$\therefore\ W = \tfrac{1}{2}(W_1 + W_2)$$

$=$ arithmetic mean between the apparent weights.

Ex. 2. *The arms of a balance are of unequal length, but the beam remains in a horizontal position when the scale-pans are loaded; shew that, if a body be placed successively in each scale-pan, its true weight is the geometrical mean between its apparent weights.*

Shew also that if a tradesman appear to weigh out equal quantities of the same substance, using alternately each of the scale-pans, he will defraud himself.

Since the beam remains horizontal when there are no weights in the scale-pans, it follows that the centre of gravity of the beam must be vertically under the fulcrum.

Let a and b be the lengths of the arms of the beam and let a body, whose true weight is W, appear to weigh W_1 and W_2 successively.

Hence $\qquad W\,.\,a = W_1\,.\,b$ ....(1),

and $\qquad W_2\,.\,a = W\,.\,b$ ....(2).

$\therefore$ by multiplication we have

$$W^2\,.\,ab = W_1 W_2\,.\,ab.$$

$$\therefore W = \sqrt{W_1\,.\,W_2},$$

i.e., the true weight is the geometrical mean between the apparent weights.

Again, if the tradesman appear to weigh out in succession quantities equal to W, he really gives his customers $W_1 + W_2$.

Now $\qquad W_1 + W_2 - 2W = W\frac{a}{b} + W\frac{b}{a} - 2W$

$$= W\frac{a^2 + b^2 - 2ab}{ab} = W\frac{(a-b)^2}{ab}.$$

Now, whatever be the values of a and b, the right-hand member of this equation is always positive, so that $W_1 + W_2$ is always $> 2W$. Hence the tradesman defrauds himself.

Numerical example. If the lengths of the arms be 11 and 12 ins. respectively, and if the nominal quantity weighed be 66 lbs. in each case, the real quantities are $\frac{11}{12}\,.\,66$ and $\frac{12}{11}\,.\,66$, *i.e.*, 60½ and 72, *i.e.*, 132½ lbs., so that the tradesman loses ½ lb.

Ex. 3. *If a balance be unjustly weighted, and have unequal arms, and if a tradesman weigh out to a customer a quantity* 2W *of some substance by weighing equal portions in the two scale-pans, shew that he will defraud himself if the centre of gravity of the beam be in the longer arm.*

Let a and b be the lengths of the arms; and let the weight W' of the machine act at a point in the arm b at a distance x from the fulcrum. Let a body of weight W, placed in the two pans in succession, be balanced by W_1 and W_2 respectively. Then we have

$$W . a = W_1 . b + W' . x,$$

and

$$W_2 . a = W . b + W' . x.$$

$$\therefore W_1 + W_2 - 2W = \frac{W . a - W' . x}{b} + \frac{W . b + W' . x}{a} - 2W$$

$$= \frac{W(b-a)^2}{ab} + W' \frac{b-a}{ab} x.$$

If b be $> a$, the right-hand member of this equation is positive, and then

$$W_1 + W_2 \text{ is } > 2W.$$

Hence, if the centre of gravity of the beam be in the longer arm, the tradesman will defraud himself.

EXAMPLES. XXVI.

1. The only fault in a balance being the unequalness in weight of the scale-pans, what is the real weight of a body which balances 10 lbs. when placed in one scale-pan, and 12 lbs. when placed in the other?

2. The arms of a balance are $8\frac{3}{4}$ and 9 ins. respectively, the goods to be weighed being suspended from the longer arm; find the real weight of goods whose apparent weight is 27 lbs.

3. One scale of a common balance is loaded so that the apparent weight of a body, whose true weight is 18 ounces, is 20 ounces; find the weight with which the scale is loaded.

4. A substance, weighed from the two arms successively of a balance, has apparent weights 9 and 4 lbs. Find the ratio of the lengths of the arms and the true weight of the body.

5. A body, when placed in one scale-pan, appears to weigh 24 lbs. and, when placed in the other, 25 lbs. Find its true weight to three places of decimals, assuming the arms of the scale-pans to be of unequal length.

6. A piece of lead in one pan A of a balance is counterpoised by 100 grains in the pan B; when the same piece of lead is put into the pan B it requires 104 grains in A to balance it; what is the ratio of the length of the arms of the balance?

7. A body, placed in a scale-pan, is balanced by 10 lbs. placed in the other pan; when the position of the body and the weights are interchanged, 11 lbs. are required to balance the body. If the length of the shorter arm be 12 ins., find the length of the longer arm and the weight of the body.

8. The arms of a false balance, whose weight is neglected, are in the ratio of 10 : 9. If goods be alternately weighed from each arm, shew that the seller loses $\frac{5}{9}$th per cent.

9. If the arms of a false balance be 8 and 9 ins. long respectively, find the prices really paid by a person for tea at two shillings per lb., if the tea be weighed out from the end of (1) the longer, (2) the shorter arm.

10. A dealer has correct weights, but one arm of his balance is $\frac{1}{20}$th part shorter than the other. If he sell two quantities of a certain drug, each apparently weighing $9\frac{1}{2}$ lbs., at 40*s*. per lb., weighing one in one scale and the other in the other, what will he gain or lose?

11. When a given balance is loaded with equal weights, it is found that the beam is not horizontal, but it is not known whether the arms are of unequal length, or the scale-pans of unequal weight; 51·075 grains in one scale balance 51·362 in the other, and 25·592 grains balance 25·879 grains; shew that the arms are equal, but that the scale-pans differ in weight by ·287 grains.

12. P and Q balance on a common balance; on interchanging them it is found that we must add to Q one-hundredth part of itself; what is the ratio of the arms and the ratio of P to Q?

13. A true balance has one scale unjustly loaded; if a body be successively weighed in the two scales and appear to weigh P and Q pounds respectively, find the amount of the unjust load and also the true weight of the body.

14. The arms of a false balance are unequal and the scale loaded; a body, whose true weight is P lbs., appears to weigh w lbs. when placed in one scale and w' lbs. when placed in the other; find the ratio of the arms and the weight with which the scale is loaded.

15. In a loaded balance with unequal arms, P appears to weigh Q, and Q appears to weigh R; find what R appears to weigh.

16. A piece of wood in the form of a long wedge, of uniform width, one end being $\frac{1}{2}$-inch and the other $\frac{1}{4}$-inch thick, is suspended by its centre of gravity and used as the beam of a balance, the goods to be weighed being suspended from the longer arm; find the true weight of goods whose apparent weight is 20 lbs.

17. The arms of a false balance are a and b, and a weight W balances P at the end of the shorter arm b, and Q at the end of the arm a; shew that

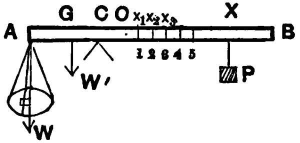
$$\frac{a}{b} = \frac{P - W}{W - Q}.$$

18. If a man, sitting in one scale of a weighing-machine, press with a stick against any point of the beam between the point from which the scale is suspended and the fulcrum, shew that he will appear to weigh more than before.

VI. The Steelyards.

164. The **Common**, or Roman, Steelyard is a machine for weighing bodies and consists of a rod, AB, movable about a fixed fulcrum at a point C.

At the point A is attached a scale-pan which contains the body to be weighed, and on the arm CB slides a movable weight P. The point at which P must be placed, in order that the beam may rest in a horizontal position, determines the weight of the body in the scale-pan. The arm CB has numbers engraved on it at different points of its length, so that the graduation at which the weight P rests gives the weight of the body.



165. *To graduate the Steelyard.* Let W' be the weight of the steelyard and the scale-pan, and let G be the point of the beam through which W' acts. The beam is usually constructed so that G lies in the shorter arm AC.

When there is no weight in the scale-pan, let O be the point in CB at which the movable weight P must be placed to balance W'.

Taking moments about C, we have

$$W' . GC = P . CO \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

This condition determines the position of the point O which is the zero of graduation.

When the weight in the scale-pan is W, let X be the point at which P must be placed. Taking moments, we have

$$W \,.\, CA + W' \,.\, GC = P \,.\, CX \;\ldots\ldots\ldots\ldots\text{(ii)}.$$

By subtracting equation (i) from equation (ii), we have

$$W \,.\, CA = P \,.\, OX.$$

$$\therefore\; OX = \frac{W}{P} \,.\, CA \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

First, let $W = P$; then, by (iii), we have

$$OX = CA.$$

Hence, if from O we measure off a distance $OX_1\,(= CA)$, and if we mark the point OX_1 with the figure 1, then, when the movable weight rests here, the body in the scale-pan is P lbs.

Secondly, let $W = 2P$; then, from (iii), $OX = 2CA$.

Hence from O mark off a distance $2CA$, and at the extremity put the figure 2. Thirdly, let $W = 3P$; then, from (iii), $OX = 3CA$, and we therefore mark off a distance from O equal to $3CA$, and mark the extremity with the figure 3.

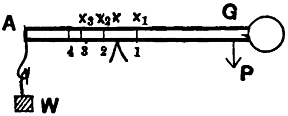
Hence, to graduate the steelyard, we must mark off from O successive distances CA, $2CA$, $3CA$,... and at their extremities put the figures 1, 2, 3, 4,.... The intermediate spaces can be subdivided to shew fractions of P lbs.

If the movable weight be 1 lb., the graduations will shew pounds.

Cor. Since the distances between successive graduations are equal, it follows that the distances of the points of graduations from the fulcrum, corresponding to equal increments of weight, form an arithmetical progression whose common difference is the distance between the fulcrum and the point at which the body to be weighed is attached.

166. When the centre of gravity G of the machine is in the longer arm, the point O from which the graduations are to be measured must lie in the shorter arm. The theory will be the same as before, except that in this case we shall have to add the equations (i) and (ii).

167. The **Danish** steelyard consists of a bar AB, terminating in a heavy knob, or ball, B. At A is attached a scale-pan in which is placed the body to be weighed.



The weight of the body is determined by observing about what point of the bar the machine balances.

[This is usually done by having a loop of string, which can slide along the bar, and finding where the loop must be to give equilibrium.]

168. *To graduate the Danish steelyard.* Let P be the weight of the bar and scale-pan, and let G be their common centre of gravity. When a body of weight W is placed in the scale-pan, let X be the position of the fulcrum.

By taking moments about X, we have

$$AX \,.\, W = XG \,.\, P = (AG - AX) \,.\, P.$$

$$\therefore\; AX\,(P + W) = P \,.\, AG.$$

$$\therefore\; AX = \frac{P}{P+W} \,.\, AG \qquad \text{..................(i).}$$

First, let $W = P$; then $AX = \frac{1}{2} AG$.

Hence bisect AG and at the middle point, X_1, engrave the figure 1; when the steelyard balances about this point the weight of the body in the scale-pan is P.

Secondly, let $W = 2P$; then $AX = \frac{1}{3} AG$.

Take a point at a distance from A equal to $\frac{1}{3}AG$ and mark it 2.

Next, let W in succession be equal to $3P$, $4P$,...; from (i), the corresponding values of AX are $\frac{1}{4}AG$, $\frac{1}{5}AG$,.... Take points of the bar at these distances from A and mark them 3, 4,....

Finally, let $W = \frac{1}{2}P$; then, from (i), $AX = \frac{2}{3}AG$;
and let $W = \frac{1}{3}P$; then, from (i), $AX = \frac{3}{4}AG$.

Take points whose distances from A are $\frac{2}{3}AG$, $\frac{3}{4}AG$, $\frac{4}{5}AG$,..., and mark them $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$,....

It will be noticed that the point G can be easily determined; for it is the position of the fulcrum when the steelyard balances without any weight in the scale-pan.

Cor. Since AX_1, AX_2, AX_3,... are inversely proportional to the numbers 2, 3, 4,... they form an harmonical progression; hence the distances of the points of graduation from the scale-pan (corresponding to equal increments of the body to be weighed) are in harmonical progression.

169. **Ex.** *A Danish steelyard weighs 6 lbs., and the distance of its centre of gravity from the scale-pan is 3 feet; find the distances of the successive points of graduation from the fulcrum.*

Taking the notation of the preceding article, we have $P=6$, and $AG=3$ feet.

$$\therefore \quad AX = \frac{6}{6+W} \times 3 = \frac{18}{W+6} \text{ feet.}$$

$$\therefore \text{ when } W=1,\ AX_1 = \tfrac{18}{7} = 2\tfrac{4}{7} \text{ feet,}$$

$$\text{when } W=2,\ AX_2 = \tfrac{18}{8} = 2\tfrac{1}{4} \text{ feet,}$$

$$\text{when } W=3,\ AX_3 = \tfrac{18}{9} = 2 \text{ feet,}$$

....

$$\text{when } W=\tfrac{1}{2},\ AX_{\frac{1}{2}} = \frac{18}{\frac{1}{2}+6} = 2\tfrac{10}{13} \text{ feet,}$$

and so on.

These give the required graduations.

EXAMPLES. XXVII.

1. A common steelyard weighs 10 lbs.; the weight is suspended from a point 4 inches from the fulcrum, and the centre of gravity of the steelyard is 3 inches on the other side of the fulcrum; the movable weight is 12 lbs.; where should the graduation corresponding to 1 cwt. be situated?

2. A heavy tapering rod, $14\frac{1}{2}$ inches long and of weight 3 lbs., has its centre of gravity $1\frac{2}{3}$ inches from the thick end and is used as a steelyard with a movable weight of 2 lbs.; where must the fulcrum be placed, so that it may weigh up to 12 lbs., and what are the intervals between the graduations that denote pounds?

3. In a steelyard, in which the distance of the fulcrum from the point of suspension of the weight is one inch and the movable weight is 6 ozs., to weigh 15 lbs. the weight must be placed 8 inches from the fulcrum; where must it be placed to weigh 24 lbs.?

4. The fulcrum is distant $1\frac{1}{3}$ inches from the point at which are suspended the goods to be weighed, and is distant 2 inches from the centre of gravity of the bar; the bar itself weighs 3 lbs. and a 2 lb. weight slides on it. At what distance apart are the graduations marking successive pounds' weight, and what is the least weight that can be weighed?

5. A steelyard, *AB*, 4 feet long, has its centre of gravity 11 inches, and its fulcrum 8 inches, from *A*. If the weight of the machine be 4 lbs. and the movable weight be 3 lbs., find how many inches from *B* is the graduation marking 15 lbs.

6. A uniform bar, *AB*, 2 feet long and weighing 3 lbs., is used as a steelyard, being supported at a point 4 inches from *A*. Find the greatest weight that can be weighed with a movable weight of 2 lbs., and find also the point from which the graduations are measured.

7. A uniform rod being divided into 20 equal parts, the fulcrum is placed at the first graduation. The greatest and least weights which the instrument can weigh are 20 and 2 lbs.; find its weight and the magnitude of the movable weight.

8. A uniform rod, 2 feet long and of weight 3 lbs., is used as a steelyard, whose fulcrum is 2 inches from one end, the sliding weight being 1 lb. Find the greatest and the least weights that can be measured.

Where should the sliding weight be to shew 20 lbs.?

9. The beam of a steelyard is 33 inches in length; the fulcrum is distant 4 inches and the centre of gravity of the beam $5\frac{1}{2}$ inches from the point of attachment of the weight; if the weight of the beam be 6 lbs. and the heaviest weight that can be weighed be 24 lbs., find the magnitude of the movable weight.

10. A steelyard is formed of a uniform bar, 3 feet long and weighing $2\frac{1}{4}$ lbs., and the fulcrum is distant 4 inches from one end; if the movable weight be 1 lb., find the greatest and least weights that can be weighed by the machine and the distance between the graduations when it is graduated to shew pounds.

11. A common steelyard, supposed uniform, is 40 inches long, the weight of the beam is equal to the movable weight, and the greatest weight that can be weighed by it is four times the movable weight; find the position of the fulcrum.

12. In a Danish steelyard the distance between the zero graduation and the end of the instrument is divided into 20 equal parts and the greatest weight that can be weighed is 3 lbs. 9 ozs.; find the weight of the instrument.

13. Find the length of a Danish steelyard, whose weight is 1 lb., and in which the distance between the graduations denoting 4 and 5 lbs. is one inch.

14. In a Danish steelyard the fulcrum rests halfway between the first and second graduation; shew that the weight in the scale-pan is $\frac{7}{5}$ths of the weight of the bar.

15. If the weight of a steelyard be worn away to one-half, its length and centre of gravity remaining unaltered, what corrections must be applied to make the weighing true, if the distance of the zero point of graduation from the fulcrum were originally one-third of the distance between successive graduations, and if the movable weight be one pound?

16. A steelyard by use loses $\frac{1}{10}$th of its weight, its centre of gravity remaining unaltered; shew how to correct its graduations.

17. A shopman, using a common steelyard, alters the movable weight for which it has been graduated; does he cheat himself or his customers?

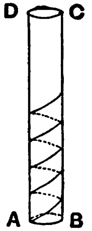
18. In a weighing machine constructed on the principle of a common steelyard, the pounds are read off by graduations reading from 0 to 14 lbs., and the stones by a weight hung at the end of the arm; if the weight corresponding to one stone be 7 ounces, the movable weight $\frac{1}{2}$ lb., and the length of the arm 1 foot, shew that the distance between successive graduations is $\frac{3}{4}$ inch.

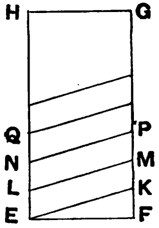
19. A weighing machine is constructed so that for each complete stone placed in the weighing pan an additional mass of m ounces has to be placed at the end of the arm, which is one foot in length, whilst the odd pounds in the weighing pan are measured by a mass of n ounces sliding along the weighing arm. Shew that the distances between the graduations for successive lbs. must be $\frac{6m}{7n}$ inches.

VII. The Screw.

170. A Screw consists of a cylinder of metal round the outside of which runs a protuberant thread of metal.

Let *ABCD* be a solid cylinder, and let *EFGH* be a





rectangle, whose base *EF* is equal to the circumference of the solid cylinder. On *EH* and *FG* take points

$$L,\ N,\ Q \ldots \quad \text{and} \quad K,\ M,\ P \ldots$$

such that $EL,\ LN, \ldots FK,\ KM,\ MP \ldots$

are all equal, and join *EK*, *LM*, *NP*, ...

Wrap the rectangle round the cylinder, so that the point *E* coincides with *A* and *EF* with the line *AB*. On being wrapped round the cylinder the point *F* will coincide with *E* at *A*.

The lines *EK*, *LM*, *NP* ... will now become a continuous spiral line on the surface of the cylinder and, if we imagine the metal along this spiral line to become protuberant, we shall have the thread of a screw.

It is evident, by the method of construction, that the thread is an inclined plane running round the cylinder and that its inclination to the horizon is the same everywhere and equal to the angle KEF. This angle is often called the angle of the screw, and the distance between two consecutive threads, measured parallel to the axis, is called the pitch of the screw.

It is clear that KF is equal to the distance between consecutive threads on the screw, and that EF is equal to circumference of the cylinder on which the thread is traced.

$$\therefore \tan(\text{angle of screw}) = \frac{KF}{EF}$$

$$= \frac{\text{pitch of screw}}{\text{circumference of a circle whose radius is the distance from the axis of any point of the screw.}}$$

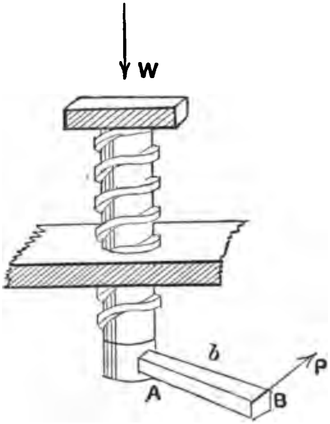
The section of the thread of the screw has, in practice, various shapes. The only kind that we shall consider has the section rectangular.

171. The screw usually works in a fixed support, along the inside of which is cut out a hollow of the same shape as the thread of the screw, and along which the thread slides. The only movement admissible to the screw is to revolve about its axis, and at the same time to move in a direction parallel to its length.

If the screw were placed in an upright position, and a weight placed on its top, the screw would revolve and descend since there is supposed to be no friction between the screw and its support. Hence, if the screw is to remain in equilibrium, some power must be applied to it; this power is usually applied at one end of a horizontal arm, the other end of which is rigidly attached to the screw.

172. *In a smooth screw, to find the relation between the power and the weight.*

Let a be the distance of any point on the thread of the screw from its axis, and b the distance, AB, from the axis of the screw, of the point at which the power is applied.



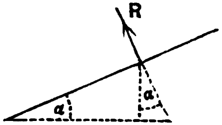
The screw is in equilibrium under the action of the power P, the weight W, and the reactions at the points in which the fixed block touches the thread of the screw. Let R, S, T, ... denote the pressures of the block at different points of the thread of the screw. These will be all perpendicular to the thread of the screw, since it is smooth.

Let α be the inclination of the thread of the screw to the horizon.

The horizontal and vertical components of the pressure R are $R \sin \alpha$ and $R \cos \alpha$ respectively.

Similarly, we may resolve S, T, ...

Hence the pressures of the block are equivalent to a set of forces $R\cos\alpha$, $S\cos\alpha$, $T\cos\alpha$, ... vertically, and a set $R\sin\alpha$, $S\sin\alpha$, $T\sin\alpha$, ... horizontally. These latter forces, though they act at different points of the screw, all act at the same distance from the axis of the screw;



they also tend to turn the screw in the opposite direction to that of P.

Equating the vertical forces, we have

$$W = R\cos\alpha + S\cos\alpha + \ldots = (R + S + T + \ldots)\cos\alpha \ldots(1).$$

Also, taking moments about the axis of the screw, we have, by Art. 93,

$$\begin{aligned} P\,.\,b &= R\sin\alpha\,.\,a + S\sin\alpha\,.\,a + T\sin\alpha\,.\,a + \ldots \\ &= a\sin\alpha\,(R + S + T + \ldots) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2). \end{aligned}$$

From equations (1) and (2) we have, by division,

$$\frac{P\,.\,b}{W} = \frac{a\sin\alpha}{\cos\alpha},$$

$$\therefore\ \frac{P}{W} = \frac{a}{b}\tan\alpha = \frac{2\pi a\tan\alpha}{2\pi b}.$$

But, by Art. 170,

$2\pi a\tan\alpha$ = distance between consecutive threads.

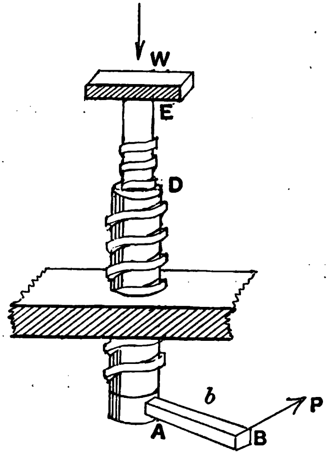
Also $2\pi b$ = circumference of the circle described by the end B of the power-arm.

Hence the mechanical advantage $= \dfrac{W}{P}$

$$= \frac{\text{circumference of a circle whose radius is the power-arm}}{\text{distance between consecutive threads of the screw}}.$$

*173. Theoretically, the mechanical advantage in the case of the screw can be made as large as we please, by decreasing sufficiently the distance between the threads of the screw. In practice, however, this is impossible; for, if we diminish the distance between the threads to too small a quantity, the threads themselves would not be sufficiently strong to bear the strain put upon them.

By means of **Hunter's Differential Screw** this difficulty may be overcome.



In this machine we have a screw AD working in a fixed block. The inside of the screw AD is hollow and is grooved to admit a smaller screw DE. The screw DE is fastened at E to a block, so that it cannot rotate, but can only move in the direction of its length.

When the power-arm AB has made one revolution, the screw AD has advanced a distance equal to the distance between two consecutive threads, and at the same time the smaller screw goes into DA a distance equal to the distance between two consecutive threads of the smaller screw. Hence the smaller screw, and therefore also the weight, advances a distance equal to the *difference* of these two distances.

When in equilibrium let R, S, T, ... be the pressures between the larger screw and its block, and R', S', T', ... the pressures between the inner and outer screws; let b and b' be the radii, and α and α' the angles of the screws.

As in the last article, since the outer screw is in equilibrium, we have

$$P \cdot b = (R + S + T + \ldots) \sin \alpha \cdot a - (R' + S' + \ldots) \sin \alpha' \cdot a' \qquad \text{.........(1)},$$

and $$(R + S + T + \ldots) \cos \alpha = (R' + S' + \ldots) \cos \alpha' \text{......(2)}.$$

Also, since the inner screw is in equilibrium, we have

$$W = (R' + S' + T' \ldots) \cos \alpha' \text{ ....(3)}.$$

From (2) and (3), we have

$$R' + S' + \ldots = \frac{W}{\cos \alpha'}, \text{ and } R + S + \ldots = \frac{W}{\cos \alpha}.$$

$\therefore$ from (1),

$$P \cdot b = W \cdot a \tan \alpha - W \cdot a' \tan \alpha'.$$

$$\therefore \frac{W}{P} = \frac{2\pi b}{2\pi a \tan \alpha - 2\pi a' \tan \alpha'}$$

$$= \frac{\text{circum. of the circle described by the end of the power-arm}}{\text{difference of the pitches of the two screws.}}.$$

By making the pitches of the two screws nearly equal, we can make the mechanical advantage very great without weakening the machine.

EXAMPLES. XXVIII.

1. Find what mass can be lifted by a smooth vertical screw of 1½ ins. pitch, if the power be a force of 25 lbs. wt. acting at the end of an arm, 3½ feet long.

2. What must be the length of the power-arm of a screw, having 6 threads to the inch, so that the mechanical efficiency may be 216?

3. What force applied to the end of an arm, 18 ins. long, will produce a pressure of 1100 lbs. wt. upon the head of a screw, when seven turns cause the screw to advance through ⅔rds of an inch?

4. A screw, whose pitch is ¼ inch, is turned by means of a lever, 4 feet long; find the power which will raise 15 cwt.

5. The arm of a screw-jack is 1 yard long, and the screw has 2 threads to the inch. What force must be applied to the arm to raise 1 ton?

6. What is the thrust caused by a screw, having 4 threads to the inch, when a force of 50 lbs. wt. is applied to the end of an arm, 2 feet long?

7. What pressure will a screw, whose arm is 2 feet and with 10 threads per foot of its length, produce, if the power be a force of 112 lbs. weight?

8. If the power be applied at the end of an arm of 1 foot in length, and if the screw make seven complete turns in 1 foot of its length, find the power that will support a weight of 1 ton.

9. If the lever by which a screw is worked be 6 feet in length, determine the distance between two successive threads of the screw, in order that the pressure of 10 lbs. wt. applied to each end of the lever may produce a pressure of 1000 lbs. wt. at the end of the screw.

10. Find the mechanical advantage in a differential screw, having 5 threads to an inch and 6 threads to the inch, the power being applied at the circumference of a wheel of diameter 4 feet.

11. Find the mechanical advantage in a differential screw, the larger screw having 8 threads to the inch and the smaller 9 threads, the length of the power-arm being 1 foot.

CHAPTER XII.

FRICTION.

174. In Art. 20 we defined smooth bodies to be bodies such that, if they be in contact, the only action between them is perpendicular to both surfaces at the point of contact. With smooth bodies, therefore, there is no force tending to prevent one body sliding over the other. If a perfectly smooth body be placed on a perfectly smooth inclined plane, there is no action between the plane and the body to prevent the latter from sliding down the plane, and hence the body will not remain at rest on the plane unless some external force be applied to it.

Practically, however, there are no bodies which are perfectly smooth; there is always *some* force between two bodies in contact to prevent one sliding upon the other. Such a force is called the force of friction.

Friction. Def. *If two bodies be in contact with one another, the property of the two bodies, by virtue of which a force is exerted between the two bodies at their point of contact to prevent one body sliding on the other, is called friction; also the force exerted is called the force of friction.*

175. Friction is a self-adjusting force; no more friction is called into play than is sufficient to prevent motion.

Let a heavy slab of iron with a plane base be placed on a horizontal table. If we attach a piece of string to some point of the body, and pull in a horizontal direction passing through the centre of gravity of the slab, a resistance is felt which prevents our moving the body; this resistance is exactly equal to the force which we exert on the body.

If we now stop pulling, the force of friction also ceases to act; for, if the force of friction did not cease to act, the body would move.

The amount of friction which can be exerted between two bodies is not, however, unlimited. If we continually increase the force which we exert on the slab, we find that finally the friction is not sufficient to overcome this force, and the body moves.

176. Friction plays an important part in the mechanical problems of ordinary life. If there were no friction between our boots and the ground, we should not be able to walk; if there were no friction between a ladder and the ground, the ladder would not rest, unless held, in any position inclined to the vertical.

177. The laws of statical friction are as follows:

Law I. *When two bodies are in contact, the direction of the friction on one of them at its point of contact is opposite to the direction in which this point of contact would commence to move.*

Law II. *The magnitude of the friction is, when there is equilibrium, just sufficient to prevent the body from moving.*

178. Suppose, in Art. 151, Case I., the plane to be rough, and that the body, instead of being supported by a power, rested freely on the plane. In this case the power P is replaced by the friction, which is therefore equal to $W \sin a$.

Ex. 1. In what direction does the force of friction act in the cases of (1) the wheel of a carriage, (2) the feet of a man who is walking?

Ex. 2. A body, of weight 30 lbs., rests on a rough horizontal plane and is acted upon by a force, equal to 10 lbs. wt., making an angle of 30° with the horizontal; shew that the force of friction is equal to about 8·66 lbs. wt.

Ex. 3. A body, resting on a rough horizontal plane, is acted on by two forces, equal respectively to 7 and 8 lbs. wt., and acting at an angle of 60°; shew that the force of friction is equal to 13 lbs. wt. in a direction making an angle $\sin^{-1}\frac{4\sqrt{3}}{13}$ with the first force.

Ex. 4. A body, of weight 40 lbs., rests on a rough plane inclined at 30° to the horizon, and is supported by (1) a force of 14 lbs. wt. acting up the plane, (2) a force of 25 lbs. acting up the plane, (3) a horizontal force equal to 20 lbs. wt., (4) a force equal to 30 lbs. wt. making an angle of 30° with the plane.

Find the force of friction in each case.

Ans. (1) 6 lbs. wt. up the plane; (2) 5 lbs. wt. down the plane; (3) 2·68 lbs. wt. up the plane; (4) 5·98 lbs. wt. down the plane.

179. The above laws hold good, in general; but the amount of friction that can be exerted is limited, and equilibrium is sometimes on the point of being destroyed.

Limiting Friction. Def. *When one body is just on the point of sliding upon another body, the equilibrium is said to be limiting, and the friction then exerted is called limiting friction.*

180. The direction of the limiting friction is given by Law I. (Art. 177).

The magnitude of the limiting friction is given by the three following laws.

Law III. *The force of friction always bears a constant ratio to the normal reaction, and this ratio depends only on the substances of which the bodies are composed.*

Law IV. *The friction is independent of the extent and shape of the surfaces in contact, so long as the normal reaction is unaltered.*

Law V. *When motion ensues, by one body sliding over the other, the direction of friction is opposite to the direction of motion; the magnitude of the friction is independent of the velocity, but the ratio of the friction to the normal reaction is slightly less than when the body is at rest and just on the point of motion.*

The above laws are experimental, and cannot be accepted as rigorously accurate; they represent, however, to a fair degree of accuracy the actual circumstances.

181. **Coefficient of friction.** The constant ratio of the friction to the normal pressure is called the coefficient of friction, and is generally denoted by μ; hence, if F be the friction, and R the normal pressure, between two bodies when equilibrium is on the point of being destroyed, we have $\frac{F}{R} = \mu$, and hence $F = \mu R$.

The values of μ are widely different for different pairs of substances in contact; no pairs of substances are, however, known for which the coefficient of friction is as great as unity.

182. **Angle of Friction.** When the equilibrium is limiting, if the friction and the normal reaction be compounded into one single force, the angle which this force makes with the normal is called the angle of friction, and the single force is called the resultant reaction.

Let A be the point of contact of the two bodies, and let AB and AC be the directions of the normal force R and the friction μR.

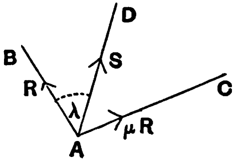
Let AD be the direction of the resultant reaction S, so that the angle of friction is BAD. Let this angle be λ.

Since R and μR are the components of S, we have

$$S\cos\lambda = R,$$

and

$$S\sin\lambda = \mu R.$$



Hence, by squaring and adding, we have

$$S = R\sqrt{1+\mu^2},$$

and, by division, $\tan\lambda = \mu$.

Hence we see that *the coefficient of friction is equal to the tangent of the angle of friction.*

183. Since the greatest value of the friction is μR, it follows that the greatest angle which the direction of resultant reaction can make with the normal is λ, *i.e.*, $\tan^{-1}\mu$.

Hence, if two bodies be in contact and if, with the common normal as axis, and the point of contact as vertex, we describe a cone whose semi-vertical angle is $\tan^{-1}\mu$, it is possible for the resultant reaction to have any direction lying within, or upon, this cone, but it cannot have any direction lying without the cone.

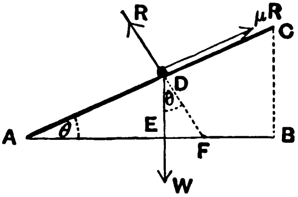
This cone is called the **Cone of friction.**

184. The following table, taken from Prof. Rankine's *Machinery and Millwork*, gives the coefficients and angles of friction for a few substances.

SUBSTANCES	μ	λ
Wood on wood —Dry	·25 to ·5	14° to 26½°
,, ,, ,, —Soaped	·04 to ·2	2° to 11½°
Metals on metals —Dry	·15 to ·2	8½° to 11½°
,, ,, ,, —Wet	·3	16½°
Leather on metals—Dry	·56	29½°
,, ,, ,, —Wet	·36	20°
,, ,, ,, —Oily	·15	8½°

185. *If a body be placed upon a rough inclined plane, and be on the point of sliding down the plane under the action of its weight and the reactions of the plane* only, *the angle of inclination of the plane to the horizon is equal to the angle of friction.*

Let θ be the inclination of the plane to the horizon, W the weight of the body, and R the normal reaction.



Since the body is on the point of motion *down* the plane, the friction acts *up* the plane and is equal to μR.

Resolving perpendicular and parallel to the plane, we have

$$W \cos\theta = R,$$

and

$$W \sin\theta = \mu R.$$

$\therefore$ by division,

$$\tan\theta = \mu = \tan\text{ (angle of friction)},$$

$$\therefore \theta = \text{the angle of friction.}$$

This may be shewn otherwise thus:

Since the body is in equilibrium under the action of its weight and the resultant reaction, the latter must be vertical; but, since the equilibrium is limiting, the resultant reaction makes with the normal the angle of friction.

Hence the angle between the normal and the vertical is the angle of friction, *i.e.*, the inclination of the plane to the horizon is the angle of friction.

N. B. The student must carefully notice that, when the body is *supported* on the inclined plane by an external force, it must not be assumed that the coefficient of friction is equal to the tangent of inclination of the plane to the horizon.

186. *To determine the coefficient of friction experimentally.*

By means of the theorem of the previous article the coefficient of friction between two bodies may be experimentally obtained.

For let an inclined plane be made of one of the substances and let its face be made as smooth as is possible; on this face let there be placed a slab, having a plane face, composed of the other substance.

If the angle of inclination of the plane be gradually increased, until the slab *just* slides, the tangent of the angle of inclination is the coefficient of friction.

To obtain the result as accurately as possible, the experiment should be performed a large number of times with the same substances, and the mean of all the results taken.

187. **Equilibrium on a rough inclined plane.** *A body is placed on a rough plane inclined to the horizon at an angle greater than the angle of friction, and is supported by a force, acting parallel to the plane, and along a line of greatest slope; to find the limits between which the force must lie.*

Let α be the inclination of the plane to the horizon, W the weight of the body, and R the normal reaction (Fig. I., Art. 151).

(i) Let the body be on the point of motion *down* the plane, so that the friction acts *up* the plane and is equal to μR; let P be the force required to keep the body at rest.

Resolving parallel and perpendicular to the plane, we have

$$P + \mu R = W \sin \alpha \quad \text{.................(1)},$$

and

$$R = W \cos \alpha \quad \text{.................(2)},$$

$$\therefore\ P = W(\sin \alpha - \mu \cos \alpha).$$

If $\mu = \tan\lambda$, we have

$$P = W[\sin\alpha - \tan\lambda\cos\alpha]$$

$$= W\left[\frac{\sin\alpha\cos\lambda - \sin\lambda\cos\alpha}{\cos\lambda}\right] = W\frac{\sin(\alpha-\lambda)}{\cos\lambda} \dots (3).$$

(ii) Let the body be on the point of motion *up* the plane, so that the friction acts *down* the plane and is equal to μR; let P_1 be the force required to keep the body at rest. In this case, we have

$$P_1 - \mu R = W\sin\alpha,$$

and

$$R = W\cos\alpha.$$

Hence $P_1 = W(\sin\alpha + \mu\cos\alpha)$

$$= W[\sin\alpha + \tan\lambda\cos\alpha] = W\frac{\sin(\alpha+\lambda)}{\cos\lambda} \dots\dots (4).$$

These values, P and P_1, are the limiting values of the force, if the body is to remain in equilibrium; if the force lie between P and P_1, the body remains in equilibrium, but is not on the point of motion in either direction.

Hence, for equilibrium, the force must lie between the values $W\dfrac{\sin(\alpha \pm \lambda)}{\cos\lambda}$.

It will be noted that the value of P_1 may be obtained from that of P by changing the sign of μ.

188. If the power P act at an angle θ with the inclined plane (as in Art. 151, Case III.), when the body is on the point of motion *down* the plane and the friction acts therefore *up* the plane, the equations of equilibrium are

$$P\cos\theta + \mu R = W\sin\alpha \dots\dots\dots\dots\dots (1),$$

$$P\sin\theta + R = W\cos\alpha \dots\dots\dots\dots\dots (2),$$

$\therefore$ multiplying (2) by μ, and subtracting, we have

$$P = W\frac{\sin\alpha - \mu\cos\alpha}{\cos\theta - \mu\sin\theta} = W\frac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}.$$

By substituting this value of P in (2), the value of R may be found.

When the body is on the point of motion *up* the plane we have, by changing the sign of μ,

$$P_1 = W\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}.$$

Cor. The force that will just be on the point of moving the body up the plane is least when

$$W\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)} \text{ is least,}$$

i.e., when $\cos(\theta-\lambda)$ is unity,

i.e., when $\theta=\lambda$.

Hence the force required to move the body up the plane will be least when it is applied in a direction making with the inclined plane an angle equal to the angle of friction.

EXAMPLES. XXIX.

1. A body, of weight 40 lbs., rests on a rough horizontal plane whose coefficient of friction is ·25; find the least force which acting horizontally would move the body.

Find also the least force which, acting at an angle $\cos^{-1}\frac{3}{5}$ with the horizontal, would move the body.

Determine the direction and magnitude of the resultant pressure of the plane in each case.

2. A heavy block with a plane base is resting on a rough horizontal plane. It is acted on by a force at an inclination of 45° to the plane, and the force is gradually increased till the block is just going to slide. If the coefficient of friction be ·5, compare the force with the weight of the block.

3. A mass of 30 lbs. is resting on a rough horizontal plane and can be just moved by a force of 10 lbs. wt. acting horizontally; find the coefficient of friction and the direction and magnitude of the resultant reaction of the plane.

4. Shew that the least force which will move a weight W along a rough horizontal plane is $W\sin\phi$, where ϕ is the angle of friction.

5. The inclination of a rough plane to the horizon is $\cos^{-1}\frac{12}{13}$; shew that, if the coefficient of friction be $\frac{1}{3}$, the least force, acting parallel to the plane, that will support 1 cwt. placed on the plane is $8\frac{8}{13}$ lbs. wt.; shew also that the force that would be on the point of moving the body up the plane is $77\frac{7}{13}$ lbs. wt.

6. The base of an inclined plane is 4 feet in length and the height is 3 feet; a force of 8 lbs., acting parallel to the plane, will just prevent a weight of 20 lbs. from sliding down; find the coefficient of friction.

7. A body, of weight 4 lbs., rests in limiting equilibrium on a rough plane whose slope is 30°; the plane being raised to a slope of 60°, find the force along the plane required to support the body.

8. A weight of 30 lbs. is just supported on a rough inclined plane, whose coefficient of friction is $\frac{3}{4}$, the height of the plane being $\frac{3}{5}$ths of its length. Shew that it will require a force of 36 lbs. wt. acting parallel to the plane just to be on the point of moving the weight up the plane.

9. A weight of 60 lbs. is on the point of motion down a rough inclined plane when supported by a force of 24 lbs. wt. acting parallel to the plane, and is on the point of motion up the plane when under the influence of a force of 36 lbs. wt. parallel to the plane; find the coefficient of friction.

10. Two inclined planes have a common vertex, and a string, passing over a small smooth pulley at the vertex, supports two equal weights. If one of the planes be rough and the other smooth, find the relation between the two angles of inclination of the two planes when the weight on the smooth plane is on the point of moving down.

11. Two unequal weights on a rough inclined plane are connected by a string which passes round a fixed pulley in the plane; find the greatest inclination of the plane consistent with the equilibrium of the weights.

12. Two equal weights are attached to the ends of a string which is laid over the top of two equally rough planes, having the same altitude and placed back to back, the angles of inclination of the planes to the horizon being 30° and 60° respectively; shew that the weights will be on the point of motion if the coefficient of friction be $2-\sqrt{3}$.

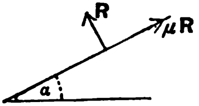
13. A particle is placed on the outside surface of a rough sphere whose coefficient of friction is μ. Shew that it will be on the point of motion when the radius from it to the centre makes an angle $\tan^{-1}\mu$ with the vertical.

14. How high can a particle rest inside a hollow sphere, of radius a, if the coefficient of friction be $\frac{1}{\sqrt{3}}$?

15. At what angle of inclination should the traces be attached to a sledge that it may be drawn up a given hill with the least exertion?

189. **Equilibrium of a rough screw.** *To find the relation between the power and the weight in the case of a screw, when friction is taken into account.*

Using the same notation as in Art. 172, let the screw be on the point of motion *downwards*, so that the friction acts *upwards* along the thread of the screw.



In this case the vertical pressures of the block are

$$R(\cos\alpha + \mu\sin\alpha),\ \ S(\cos\alpha + \mu\sin\alpha), \ldots$$

and the horizontal components of these pressures are

$$R(\sin\alpha - \mu\cos\alpha),\ \ S(\sin\alpha - \mu\cos\alpha), \ldots$$

Hence the equations (1) and (2) of Art. 172 become

$$W = (R + S + T + \ldots)(\cos\alpha + \mu\sin\alpha) \ \ldots\ldots(1).$$

$$P\,.\,b = a(R + S + T + \ldots)(\sin\alpha - \mu\cos\alpha) \ \ldots\ldots(2).$$

Hence, by division,

$$\frac{P\,.\,b}{W} = a\,\frac{\sin\alpha - \mu\cos\alpha}{\cos\alpha + \mu\sin\alpha} = a\,\frac{\sin\alpha\cos\lambda - \cos\alpha\sin\lambda}{\cos\alpha\cos\lambda + \sin\alpha\sin\lambda}$$

$$= a\,\frac{\sin(\alpha - \lambda)}{\cos(\alpha - \lambda)},$$

$$\therefore\ \ \frac{P}{W} = \frac{a}{b}\tan(\alpha - \lambda).$$

Similarly, if the screw be on the point of motion upwards, we have, by changing the sign of μ,

$$\frac{P_1}{W} = \frac{a}{b}\,\frac{\sin\alpha + \mu\cos\alpha}{\cos\alpha - \mu\sin\alpha} = \frac{a}{b}\tan(\alpha + \lambda).$$

If the power have any value between P and P_1, the screw will be in equilibrium, but the friction will not be limiting friction.

It will be noted that if the angle a of the screw be equal to the angle of friction, λ, then the value of the power P is zero. In this case the screw will just remain in equilibrium supported only by the friction along the thread of the screw.

Ex. *If the circumference of a screw be two inches, the distance between its threads half an inch, and the coefficient of friction* $\frac{1}{5}$, *find the limits between which the power must lie, so that the screw may be in equilibrium when it is supporting a body of weight* 1 *cwt., the length of the power arm being* 12 *inches.*

Here $$2\pi a = 2, \text{ and } 2\pi a \tan a = \tfrac{1}{2}.$$

$$\therefore a = \frac{1}{\pi}, \text{ and } \tan a = \tfrac{1}{4}.$$

Also $$\tan\lambda = \frac{1}{5}, \text{ and } b = 12.$$

$\therefore$ force which would *just* support the screw $= 112 \times \frac{a}{b} \tan(a - \lambda)$

$$= 112 \times \frac{1}{12\pi} \times \frac{\frac{1}{4} - \frac{1}{5}}{1 + \frac{1}{4} \cdot \frac{1}{5}} = \frac{112}{12\pi} \times \frac{1}{21} = \frac{14}{99} \text{ lbs. wt.} = \cdot\dot{1}\dot{4} \text{ lbs. wt.}$$

Again, the force which would just be on the point of moving the screw upwards

$$= 112 \times \frac{a}{b} \tan(a + \lambda) = \frac{112}{12\pi} \times \frac{\frac{1}{4} + \frac{1}{5}}{1 - \frac{1}{4} \cdot \frac{1}{5}} = \frac{112}{12\pi} \times \frac{9}{19}$$

$$= 1\tfrac{85}{209} \text{ lbs. wt.} = 1\cdot4067 \text{ lbs. wt.}$$

Hence the screw will be in equilibrium if the power lie between $\cdot\dot{1}\dot{4}$ and $1\cdot4067$ lbs. wt.

If the screw were smooth, the force required would

$$= 112 \frac{a}{b} \tan a = \frac{112}{12\pi} \times \tfrac{1}{4} = \frac{49}{66} = \cdot742 \text{ lbs. wt.}$$

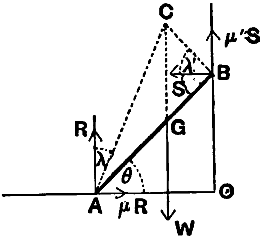
Ex. 1. The coefficient of friction of wrought iron on wood being ·15, shew that the least angle of inclination of the thread of a screw, so that it may slide into a prepared hole in the wood under the influence of its own weight, is $\tan^{-1}\frac{3}{20}$.

Ex. 2. If the circumference of a screw be $\frac{3}{2}$ inch, the coefficient of friction ·15, the length of the power arm 12 inches, and if there be 3 threads to the inch, find the forces which will respectively just support, and just move, the screw when it supports a weight W. Find also the value of the power when the same screw is smooth.

Ans. $\frac{W}{16\pi}(\cdot07)$; $\frac{W}{16\pi}(\cdot385)$; $\frac{W}{16\pi}(\cdot\dot{2})$.

190. **Ex.** *A uniform ladder is in equilibrium, with one end resting on the ground, and the other end against a vertical wall; if the ground and wall be both rough, the coefficients of friction being μ and μ' respectively, and if the ladder be on the point of slipping at both ends, find the inclination of the ladder to the horizon.*

Let AB be the ladder, and G its centre of gravity; let R and S be the normal reactions at A and B respectively; the end A of the ladder is on the point of slipping *from* the wall, and hence the friction μR is *towards* the wall; the end B is on the point of motion vertically *downwards*, and therefore the friction $\mu' S$ acts *upwards.*



Let θ be the inclination of the ladder to the wall, and $2a$ its length.

Resolving horizontally and vertically, we have

$$\mu R = S \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$R + \mu' S = W \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Also, taking moments about A, we have

$$W \, . \, a \cos\theta = \mu' S \, . \, 2a \cos\theta + S \, . \, 2a \sin\theta,$$

$$\therefore \; W \cos\theta = 2S(\mu' \cos\theta + \sin\theta) \ldots\ldots\ldots(3).$$

From (1) and (2), we have

$$\mu(W - \mu' S) = S$$

and

$$\therefore \; \mu W = S(1 + \mu\mu') \ldots\ldots\ldots\ldots\ldots(4).$$

By (3) and (4), we have, by division,

$$\frac{\cos\theta}{\mu} = \frac{2(\mu'\cos\theta + \sin\theta)}{1+\mu\mu'},$$

$$\therefore \cos\theta(1-\mu\mu') = 2\mu\sin\theta.$$

Hence $$\tan\theta = \frac{1-\mu\mu'}{2\mu}.$$

Otherwise thus;

Let λ, λ' be the angles of friction at A and B; draw AC making an angle λ with the normal at A, and BC making an angle λ' with the normal at B, as in the figure.

By Art. 182, AC and BC are the directions of the resultant reactions at A and B.

The ladder is kept in equilibrium by these resultant reactions and its weight; hence their directions must meet in a point and therefore the vertical line through G must pass through C.

From the triangles AGC, GCB we have

$$\frac{AG}{GC} = \frac{\sin GCA}{\sin CAG} = \frac{\sin\lambda}{\cos(\lambda+\theta)},$$

and $$\frac{BG}{GC} = \frac{\sin GCB}{\sin CBG} = \frac{\cos\lambda'}{\sin(\lambda'+\theta)}.$$

But $AG = BG$; hence

$$\frac{\sin\lambda}{\cos(\lambda+\theta)} = \frac{\cos\lambda'}{\sin(\lambda'+\theta)}.$$

$$\therefore \sin\lambda(\sin\lambda'\cos\theta + \cos\lambda'\sin\theta) = \cos\lambda'(\cos\lambda\cos\theta - \sin\lambda\sin\theta).$$

Hence, by dividing by $\cos\lambda\cos\lambda'\cos\theta$, we have

$$\tan\theta = \frac{1-\tan\lambda\tan\lambda'}{2\tan\lambda} = \frac{1-\mu\mu'}{2\mu}.$$

EXAMPLES. XXX.

1. A uniform ladder, 13 feet long, rests with one end against a smooth vertical wall and the other on a rough horizontal plane at a point 5 feet from the wall; find the friction between the ladder and the ground, if the weight of the ladder be 56 lbs.

2. A uniform ladder rests with one end on a horizontal floor and the other against a vertical wall, the coefficients of friction being respectively $\frac{3}{7}$ and $\frac{1}{3}$; find the inclination of the ladder when it is about to slip.

3. If in the last example the coefficient of friction in each case be $\frac{1}{3}$, shew that the ladder will slip when its inclination to the vertical is $\tan^{-1}\frac{3}{4}$.

4. A uniform ladder rests in limiting equilibrium with one end on a rough floor, whose coefficient of friction is μ, and with the other against a smooth vertical wall; shew that its inclination to the vertical is $\tan^{-1}(2\mu)$.

5. A uniform ladder is placed against a wall; if the ground and wall be equally rough, the coefficient of friction being $\tan\theta$, shew that the limiting inclination of the ladder to the vertical is 2θ.

When the ladder is in this position can it be ascended without its slipping?

6. A uniform ladder rests in limiting equilibrium with one end on a rough horizontal plane, and the other against a smooth vertical wall; a man standing on the ladder ascends it; shew that he cannot go more than half way up.

7. A uniform ladder rests with one end against a smooth vertical wall and the other on the ground, the coefficient of friction being $\frac{3}{4}$; if the inclination of the ladder to the ground be 45°, shew that a man, whose weight is equal to that of the ladder, can just ascend to the top of the ladder without its slipping.

8. A uniform ladder, of length 70 feet, rests against a vertical wall with which it makes an angle of 45°, the coefficients of friction between the ladder and the wall and ground respectively being $\frac{1}{3}$ and $\frac{1}{2}$. If a man, whose weight is one-half that of the ladder, ascend the ladder, how high will he be when the ladder slips?

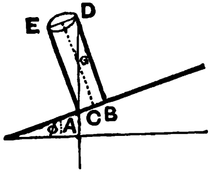
9. Two equal ladders, of weight w, are placed so as to lean against each other with their ends resting on a rough horizontal floor; given the coefficient of friction, μ, and the angle, $2a$, that they make with each other, find what weight on the top would cause them to slip.

10. A uniform ladder rests, at an angle of 45° with the horizon, with its upper extremity against a rough vertical wall and its lower extremity on the ground. If μ, μ' be the coefficients of limiting friction between the ladder and the ground and wall respectively, shew that the least horizontal force which will move the lower extremity towards the wall is $\frac{1}{2}W \cdot \dfrac{1+2\mu-\mu\mu'}{1-\mu'}$.

191. **Ex.** *A uniform cylinder is placed with its plane base on a rough inclined plane and the inclination of the plane to the horizon is gradually increased; shew that the*

cylinder will topple over before it slides if the ratio of the diameter of the base of the cylinder to its height be less than the coefficient of friction.

Let ϕ be the inclination of the plane to the horizon when the cylinder is on the point of tumbling over. The vertical line through the centre of gravity G of the cylinder must just fall within the base.



Hence, if AB be the base, the line GA must be vertical.

Let C be the middle point of the base, r its radius, and let h be the height of the cylinder,

$$\therefore \tan\phi = \cot CAG = \frac{AC}{CG} = \frac{r}{\frac{1}{2}h} = \frac{2r}{h} \text{ ....(1).}$$

Also the inclination θ of the plane to the horizon, when the cylinder is about to slide, is given by

$$\tan\theta = \mu \text{ ....(2).}$$

Hence the cylinder will topple before it slides if ϕ be less than θ,

$$\text{i.e., if } \frac{2r}{h} \text{ be } < \mu.$$

EXAMPLES. XXXI.

1. A cylinder rests with its circular base on a rough inclined plane, the coefficient of friction being $\frac{1}{2}$. Find the inclination of the plane and the relation between the height and diameter of the base of the cylinder, so that it may be on the point of sliding and also of toppling over.

2. A solid cylinder rests on a rough horizontal plane with one of its flat ends on the plane, and is acted on by a horizontal force through the centre of its upper end; if this force be just sufficient to move the solid, shew that it will slide, and not topple over, if the coefficient of friction be less than the ratio of the radius of the base of the cylinder to its height.

3. A right cylinder, of height h and radius of base a, rests with its base on a rough horizontal plane and is put in motion by a gradually increasing horizontal force applied at any point; find the limiting height of the point of application of the force, so that the initial motion may be one of sliding, and not of toppling over.

4. A conical sugarloaf, whose height is equal to twice the diameter of its base, stands on a table rough enough to prevent sliding; one end is gently raised till the sugarloaf is on the point of falling over; find the inclination of the plane to the horizon in this position.

5. A cone, of given vertical angle $2a$, rests on a rough plane which is inclined to the horizon. As the inclination of the plane is increased, shew that the cone will slide, before it topples over, if the coefficient of friction be less than $4 \tan a$.

6. A right cone is placed with its base on a rough inclined plane; if $\frac{1}{\sqrt{3}}$ be the coefficient of friction, find the angle of the cone when it is on the point of both slipping and turning over.

7. A cone rests on a rough table, and a cord fastened to the vertex of the cone passes over a smooth pulley at the same height as the top of the cone, and supports a weight. Shew that, if the weight be continually increased, the cone will turn over, or slide, according as the coefficient of friction is $>$ or $<$ $\tan a$, where a is the semi-vertical angle of the cone.

8. A cubical block rests on a rough inclined plane with its edges parallel to the edges of the plank. If, as the plank is gradually raised, the block turn on it before slipping, what is the least value that the coefficient of friction can have?

9. The triangular lamina ABC, right-angled at B, stands with BC upon a rough horizontal plane. If the plane be gradually tilted round an axis in its own plane perpendicular to BC, with the angle B downwards, shew that it will begin to slide, or topple over, according as the coefficient of friction is less, or greater, than $\tan A$.

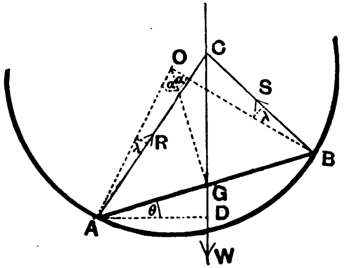
10. A square uniform metallic plate $ABCD$ rests with its side BC on a perfectly rough plane inclined to the horizon at an angle a. A string AP attached to A, the highest point of the plate, and passing over a smooth pulley at P, the vertex of the plane, supports a weight w, and AP is horizontal. If W be the weight of the plate, shew that, as w increases, it will begin to turn when

$$w > W \frac{1 + \tan a}{2}.$$

192. **Ex.** *A uniform rod rests in limiting equilibrium within a rough hollow sphere; if the rod subtend an angle $2a$ at the centre of the sphere, and if λ be the angle of friction, shew that the angle of inclination of the rod to the horizon is* $\tan^{-1}\left(\dfrac{\sin 2\lambda}{\cos 2a + \cos 2\lambda}\right)$.

Let AB be the rod, G its centre of gravity, and O the centre of the sphere, so that

$$\angle GOA = \angle GOB = a.$$



Through A and B draw lines AC and BC making an angle λ with the lines joining A and B to the centre. By Art. 182, these are the directions of the resultant reactions, R and S, at A and B respectively.

Since these reactions and the weight keep the rod in equilibrium, the vertical line through G must pass through C.

Hence we have

$$\frac{\sin ACG}{\sin GAC} = \frac{AG}{GC} = \frac{BG}{GC} = \frac{\sin GCB}{\sin GBC} \ldots\ldots\ldots\ldots (1).$$

Let AD be the horizontal line drawn through A to meet CG in D, so that the angle GAD is θ.

Then

$$\angle ACG = \frac{\pi}{2} - CAD = \frac{\pi}{2} - \left(\theta + \frac{\pi}{2} - \alpha - \lambda\right) = \alpha + \lambda - \theta;$$

also $$\angle GAC = \frac{\pi}{2} - \alpha - \lambda,$$

and $$\angle GCB = \pi - CGB - CBG$$

$$= \pi - \left(\frac{\pi}{2} - \theta\right) - \left(\lambda + \frac{\pi}{2} - \alpha\right) = \alpha - \lambda + \theta,$$

and $$\angle GBC = \lambda + \frac{\pi}{2} - \alpha = \frac{\pi}{2} - (\alpha - \lambda).$$

Hence equation (1) becomes

$$\frac{\sin(\alpha + \lambda - \theta)}{\cos(\alpha + \lambda)} = \frac{\sin(\alpha - \lambda + \theta)}{\cos(\alpha - \lambda)}.$$

$$\therefore \tan(\alpha + \lambda) - \tan\theta = \tan(\alpha - \lambda) + \tan\theta.$$

$$\therefore 2\tan\theta = \tan(\alpha + \lambda) - \tan(\alpha - \lambda) \quad \text{..........(2)}$$

$$= \frac{\sin 2\lambda}{\cos(\alpha + \lambda)\cos(\alpha - \lambda)}.$$

$$\therefore \tan\theta = \frac{\sin 2\lambda}{\cos 2\alpha + \cos 2\lambda}.$$

Otherwise thus; The solution may be also obtained by using the conditions of Art. 83.

Resolving the forces along the rod, we have

$$R\cos\left(\frac{\pi}{2} - \alpha - \lambda\right) - S\cos\left(\frac{\pi}{2} - \alpha + \lambda\right) = W\sin\theta,$$

i.e. $$R\sin(\alpha + \lambda) - S\sin(\alpha - \lambda) = W\sin\theta \quad \text{........(3).}$$

Resolving perpendicular to the rod, we have

$$R\cos(\alpha + \lambda) + S\cos(\alpha - \lambda) = W\cos\theta \quad \text{......(4).}$$

By taking moments about A, we have

$$S \cdot AB\sin\left(\frac{\pi}{2} - \alpha + \lambda\right) = W \cdot AG\cos\theta,$$

i.e. $$2S\cos(\alpha - \lambda) = W\cos\theta \quad \text{..............(5).}$$

From equations (4) and (5), we have

$$R\cos(\alpha+\lambda) = S\cos(\alpha-\lambda) = \tfrac{1}{2}W\cos\theta.$$

Substituting these values of R and S in (3), we have

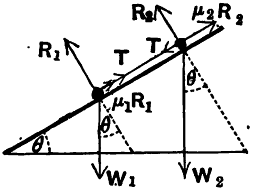
$$\tan(\alpha+\lambda) - \tan(\alpha-\lambda) = 2\tan\theta,$$

which is equation (2).

Numerical example. If the rod subtend a right angle at the centre of the sphere, shew that its inclination to the horizon is twice the angle of friction.

193. **Ex.** *Two bodies, of weights W_1 and W_2, are placed on an inclined plane and are connected by a light string which coincides with a line of greatest slope of the plane; if the coefficients of friction between the bodies and the plane be respectively μ_1 and μ_2, find the inclination of the plane to the horizon when both bodies are on the point of motion, it being assumed that the smoother body is below the other.*

Let θ be the required inclination of the plane and



R_1, R_2 the normal reactions of the bodies; also let T be the tension of the string.

The frictions $\mu_1 R_1$ and $\mu_2 R_2$ both act up the plane.

For the equilibrium of W_1, we have

$$W_1\sin\theta = T + \mu_1 R_1,$$

and

$$W_1\cos\theta = R_1.$$

$$\therefore\ T = W_1(\sin\theta - \mu_1\cos\theta) \quad\text{............(1)}.$$

For the equilibrium of W_2, we have

$$W_2 \sin\theta + T = \mu_2 R_2,$$

and $$W_2 \cos\theta \quad = R_2.$$

$$\therefore\ T = \mu_2 R_2 - W_2 \sin\theta = W_2(\mu_2 \cos\theta - \sin\theta) \ldots (2).$$

Hence, from (1) and (2),

$$W_1(\sin\theta - \mu_1 \cos\theta) = W_2(\mu_2 \cos\theta - \sin\theta).$$

$$\therefore\ (W_1 + W_2)\sin\theta = (W_1\mu_1 + W_2\mu_2)\cos\theta.$$

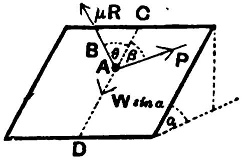
$$\therefore\ \tan\theta = \frac{W_1\mu_1 + W_2\mu_2}{W_1 + W_2}.$$

Ex. 1. Two equal bodies are placed on a rough inclined plane, being connected by a light string; if the coefficients of friction be respectively $\frac{1}{2}$ and $\frac{1}{3}$, shew that they will both be on the point of motion when the inclination of the plane is $\tan^{-1}\frac{5}{12}$.

Ex. 2. Shew that the greatest angle at which a plane may be inclined to the horizon so that three equal bodies, whose coefficients of friction are $\frac{1}{2}$, $\frac{5}{8}$, and $\frac{3}{8}$ respectively, when rigidly connected together, may rest on it without slipping, is $\tan^{-1}\frac{1}{2}$.

194. **Ex.** *A particle is placed on a rough plane, whose inclination to the horizon is α, and is acted upon by a force P acting parallel to the plane and in a direction making an angle β with the line of greatest slope in the plane; if the coefficient of friction be μ, find the direction in which the body will begin to move.*

Let W be the weight of the particle, and R the normal reaction.



The forces perpendicular to the inclined plane must vanish.

$$\therefore\ R = W\cos\alpha \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The other component of the weight will be $W \sin a$, acting down the line of greatest slope.

Let the friction, μR, act in the direction AB, making an angle θ with the line of greatest slope, so that the particle would begin to move in the direction BA produced.

Resolving the forces along the surface of the plane in directions along and perpendicular to the line of greatest slope, we have

$$P \cos \beta + \mu R \cos \theta = W \sin a \quad \text{............(2)},$$

and

$$P \sin \beta = \mu R \sin \theta \quad \text{....................(3)}.$$

From (1) substituting for R, these equations become

$$P \cos \beta = W (\sin a - \mu \cos a \cos \theta),$$

and

$$P \sin \beta = \mu W \cos a \sin \theta.$$

Hence, by division,

$$\tan \beta = \frac{\mu \cos a \sin \theta}{\sin a - \mu \cos a \cos \theta}.$$

$$\therefore \sin \beta (\sin a - \mu \cos a \cos \theta) = \mu \cos a \sin \theta \cos \beta.$$

$$\therefore \sin a \sin \beta = \mu \cos a \sin (\theta + \beta).$$

Hence

$$\sin (\theta + \beta) = \frac{\tan a \sin \beta}{\mu} \quad \text{.................(4)},$$

giving the angle θ.

Numerical Example. Suppose the inclination of the plane to be 30°, the coefficient of friction to be $\frac{1}{3}$, and the angle between the force P and the line of greatest slope to be 30°.

In this case we have

$$\sin (\theta + 30^\circ) = \frac{\tan 30^\circ \,.\, \sin 30^\circ}{\frac{1}{3}} = \frac{\sqrt{3}}{2} = \sin 60^\circ.$$

Hence θ is 30°, and the body begins to slide down the plane in a direction making an angle of 30° with the line of greatest slope.

The force P could be easily shewn to be $\dfrac{W}{\sqrt{12}}$.

*EXAMPLES. XXXII.

1. A ladder, whose centre of gravity divides it into two portions of length a and b, rests with one end on a rough horizontal floor, and the other end against a rough vertical wall. If the coefficients of friction at the floor and wall be respectively μ and μ', shew that the inclination of the ladder to the floor, when the equilibrium is limiting, is

$$\tan^{-1}\frac{a-b\mu\mu'}{\mu(a+b)}.$$

2. A weightless rod is supported horizontally between two rough inclined planes at right angles to each other, the angle of friction λ being less than the inclination of either plane. Shew that the length of that portion of the rod on which a weight may be placed without producing motion is $\sin 2\alpha \,.\, \sin 2\lambda$ of the whole length of the rod, where α is the inclination of either plane to the horizon.

3. A heavy uniform rod is placed over one and under the other of two horizontal pegs, so that the rod lies in a vertical plane; shew that the length of the shortest rod which will rest in such a position is

$$a(1+\tan\alpha\cot\lambda),$$

where a is the distance between the pegs, α is the angle of inclination to the horizon of the line joining them, and λ is the angle of friction.

4. A uniform heavy rod, 1 foot long, one end of which is rough and the other smooth, rests within a circular hoop in a vertical plane, the radius of the hoop being 10 inches. If the rod is in limiting equilibrium when its rough end is at the lowest point of the hoop, shew that the coefficient of friction is $\frac{24}{43}$.

5. A heavy uniform rod rests with its extremities on a rough circular hoop, fixed in a vertical plane; the rod subtends an angle of 120° at the centre of the hoop, and in the limiting position of equilibrium is inclined to the horizon at an angle θ. If $\sqrt{3}\mu=\tan\alpha$, μ being the coefficient of friction, shew that

$$\tan\theta : \tan 2\alpha :: 2 : \sqrt{3}.$$

6. A and B are two small equal heavy rings which slide on a rough horizontal rod, the coefficient of friction being $3^{-\frac{3}{2}}$. Another equal heavy ring C slides on a weightless smooth string connecting A and B; shew that, in the position of limiting equilibrium, ABC is an equilateral triangle.

7. One end of a heavy rod AB can slide along a rough horizontal rod AC, to which it is attached by a ring; B and C are joined by a string. If ABC be a right angle when the rod is on the point of sliding, μ the coefficient of friction, and α the angle between AB and the vertical, shew that

$$\mu = \frac{\tan\alpha}{\tan^2\alpha + 2}.$$

8. A uniform rod slides with its ends on two fixed equally rough rods, one being vertical and the other inclined at an angle α to the horizon. Shew that the inclination θ to the horizon of the movable rod, when it is on the point of sliding, is given by

$$\tan\theta = \frac{1 - 2\mu\tan\alpha - \mu^2}{2(\mu + \tan\alpha)}.$$

9. A uniform ladder, whose length is a and whose weight is W, makes an angle θ with the horizontal, and rests with one end against a vertical wall and the other upon a horizontal floor, the wall and floor being equally rough, and the coefficient of friction being $\tan\lambda$. Shew that a man, whose weight is P, can never get nearer to the top of the ladder than $\dfrac{W\cot 2\lambda + P\cot\lambda - (W+P)\tan\theta}{2P}\, a\sin 2\lambda$.

10. The poles supporting a lawn-tennis net are kept in a vertical position by guy ropes, one to each pole, which pass round pegs 2 feet distant from the poles. If the coefficient of limiting friction between the ropes and pegs be $\frac{4}{3}$, shew that the inclination of the latter to the vertical must not be less than $\tan^{-1}\frac{2}{11}$, the height of the poles being 4 feet.

11. A chest in the form of a rectangular parallelopiped, whose weight without the lid is 200 lbs., and width from back to front 1 foot, has a lid weighing 50 lbs. and stands with its back 6 inches from a smooth wall and parallel to it. If the lid be open and lean against the wall, find the least coefficient of friction between the chest and the ground that there may be no motion.

12. A heavy circular disc, whose plane is vertical, is kept at rest on a rough inclined plane by a string parallel to the plane and touching the circle. Shew that the disc will slip on the plane if the coefficient of friction be less than $\frac{1}{2}\tan i$, where i is the slope of the plane.

13. A particle resting on a rough table, whose coefficient of friction is μ, is fastened by a string, of length a, to a fixed point A on the table. Another string is fastened to the particle and, after passing over the smooth edge of the table, supports an equal particle hanging freely. Shew that the particle on the table will rest at any point of the circle, whose centre is A and whose radius is a, in which the string is kept taut and the distance of the second string from A is not greater than μa.

14. A heavy beam, of length $2a$, lies over a rough peg with one extremity leaning against a rough vertical wall; if c be the distance of the peg from the wall and λ be the angle of friction both at the peg and the wall, shew that the inclination of the rod to the wall is

$$\sin^{-1}\sqrt[3]{\frac{c}{a}\cos^2\lambda}.$$

15. A circular disc, of radius a and weight W, is placed within a smooth sphere, of radius b, and a particle, of weight w, is placed on the disc. If the coefficient of friction between the particle and the disc be μ, find the greatest distance from the centre of the disc at which the particle can rest.

16. A sphere, of given weight W, rests between a vertical wall and a prism, one of whose faces rests on a horizontal plane; if the coefficient of friction between the horizontal plane and the prism be μ, shew that the least weight of the prism consistent with equilibrium is $W\left(\frac{\tan\alpha}{\mu}-1\right)$, where α is the inclination to the horizon of the face in contact with the sphere.

17. Two equal rods, of length $2a$, are fastened together so as to form two sides of a square, and one of them rests on a rough peg. Shew that the distance of the point of contact from the middle point of the rod is $\frac{a}{2}(1-\mu)$, where μ is the coefficient of friction.

18. Two uniform rods, AC and BC, are rigidly joined at C so that they form one uniform bent rod, whose two portions are at right angles. This bent rod is supported on the edge of a rough table which touches AC at its middle point. If BC be three times AC, shew that the tangent of the inclination of AC to the horizon is $\frac{1}{3}$.

Find also the least value of the coefficient of friction that the rod may rest with the point A on the edge of the table.

19. A heavy string rests on two given inclined planes, of the same material, passing over a small pulley at their common vertex. If the string be on the point of motion, shew that the line joining its two ends is inclined to the horizon at the angle of friction.

20. On a rough inclined plane ($\mu=\frac{1}{2}$) a weight W is just supported by a force $\frac{W}{2}$ acting up the plane and parallel to it. Find the magnitude and direction of the least additional force, acting along the plane, which will prevent motion when the force $\frac{W}{2}$ acts along the plane, but at 60° with the line of greatest slope.

21. A weight W is laid upon a rough plane $\left(\mu=\frac{1}{\sqrt{3}}\right)$, inclined at 45° to the horizon, and is connected by a string passing through a smooth ring, A, at the top of the plane, with a weight P hanging vertically. If $W = 3P$, shew that, if θ be the greatest possible inclination of the string AW to the line of greatest slope in the plane, then

$$\cos\theta = \frac{2\sqrt{2}}{3}.$$

Find also the direction in which W tends to move.

22. A weight W rests on a rough inclined plane inclined at an angle α to the horizon, and the coefficient of friction is $2\tan\alpha$. Shew that the least horizontal force along the plane which will move the body is $\sqrt{3}\,W\sin\alpha$, and that the body will begin to move in a direction inclined at 60° to the line of greatest slope on the plane.

23. If two equal weights, unequally rough, be connected by a light rigid rod and be placed on an inclined plane whose inclination, α, to the horizon is the angle whose tangent is the geometric mean between the coefficients of friction, shew that the greatest possible inclination to the line of greatest slope which the rod can make when at rest is $\cos^{-1}\left(\dfrac{\mu_1+\mu_2}{2\sqrt{2\mu_1\mu_2}}\right)$, where μ_1 and μ_2 are the coefficients of friction.

24. A heavy particle is placed on a rough plane inclined at an angle α to the horizon, and is connected by a stretched weightless string AP to a fixed point A in the plane. If AB be the line of greatest slope and θ the angle PAB when the particle is on the point of slipping, shew that $\sin\theta = \mu\cot\alpha$.

Interpret the result when $\mu\cot\alpha$ is greater than unity.

25. A hemispherical shell rests on a rough plane, whose inclination to the horizon is α; shew that the inclination of the plane base of the rim to the horizon cannot be greater than $\sin^{-1}(2\sin\lambda)$.

26. A hemisphere rests on a rough horizontal plane and against a smooth vertical wall. Shew that, if the coefficient of friction be greater than $\frac{3}{8}$, the hemisphere can rest in any position and, if it be less, the greatest angle that the base of the hemisphere can make with the vertical is $\cos^{-1}\dfrac{8\mu}{3}$.

27. A heavy homogeneous hemisphere rests with its convex surface in contact with a rough inclined plane; shew that the greatest possible inclination of the plane to the horizon is $\sin^{-1}\frac{3}{8}$.

28. If a hemisphere rest in equilibrium with its curved surface in contact with a rough plane inclined to the horizon at an angle $\sin^{-1}\frac{3}{16}$, find the inclination of the plane base of the hemisphere to the vertical.

29. A uniform hemisphere, of radius a and weight W, rests with its spherical surface on a horizontal plane, and a rough particle, of weight W', rests on the plane surface; shew that the distance of the particle from the centre of the plane face is not greater than $\frac{3W\mu a}{8W'}$, where μ is the coefficient of friction.

30. A sphere, whose radius is a and whose centre of gravity is at a distance c from the centre, rests in limiting equilibrium on a rough plane inclined at an angle α to the horizon; shew that it may be turned through an angle

$$2\cos^{-1}\left(\frac{a\sin\alpha}{c}\right),$$

and still be in limiting equilibrium.

CHAPTER XIII.

WORK.

195. **Work. Def.** A force is said to do work when its point of application moves in the direction of the force.

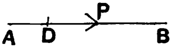
The force exerted by a horse, in dragging a waggon, does work.

The force, exerted by a man, in raising a weight, does work.

The pressure of the steam, in moving the piston of an engine, does work.

The measure of the work done by a force is the product of the force and the distance through which it moves its point of application in the direction of the force.

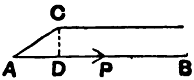
Suppose that a force acting at a point A of a body moves the point A to D, then the work done by P is measured by the product of P and AD.

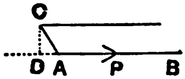


If the point D be on the side of A toward which the force acts, this work is positive; if D lie on the opposite side, the work is negative.

Next, suppose that the point of application of the force is moved to a point C, which does not lie on the line AB. Draw CD perpendicular to AB, or AB produced. Then

AD is the distance through which the point of application is moved in the direction of the force. Hence in the first figure the work done is $P \times AD$; in the second figure the work done is $-P \times AD$. When the work done by the force is negative, this is sometimes expressed by saying that the force has work done against it.





196. In the case when AC is at right angles to AB, the points A and D coincide, and the work done by the force P vanishes.

As an example, if a body be moved about on a horizontal table the work done by its weight is zero. So, again, if a body be moved on an inclined plane, no work is done by the normal reaction of the plane.

197. The unit of work, used in Statics, is called a Foot-Pound, and is the work done by a force, equal to the weight of a pound, when it moves its point of application through one foot in its own direction. A better, though more clumsy, term than "Foot-Pound" would be Foot-Pound weight.

Thus, the work done by the weight of a body of 10 pounds, whilst the body falls through a distance of 4 feet, is 10 × 4 foot-pounds.

The work done by the weight of the body, if it were raised through a vertical distance of 4 feet, would be -10×4 foot-pounds.

198. It will be noticed that the definition of work, given in Art. 195, necessarily implies motion. A man

may use great exertion in *attempting* to move a body, and yet do no work on the body.

For example, suppose a man pulls at the shafts of a heavily-loaded van, which he cannot move. He may pull to the utmost of his power, but, since the force which he exerts does not move its point of application, he does no work.

199. *To find the work done in dragging a body up a smooth inclined plane.*

Taking the figure of Art. 151, Case I., the work done by the force P in dragging the body from A to C is

$$P \times AC.$$

But $$P = W \sin \alpha.$$

$\therefore$ the work done is $W \sin \alpha \times AC$,

$$i.e., \quad W \times AC \sin \alpha, \quad i.e., \quad W \times BC.$$

Hence the work done is the same as that which would be done in lifting the weight of the body through the same height without the intervention of the inclined plane.

200. The previous article is one of the simplest examples of what we shall find to be a universal principle, viz., *Whatever be the machine we use, provided that there be no friction and that the weight of the machine be neglected, the work done by the power is always equivalent to the work done against the weight.*

Assuming that the machine we are using gives mechanical advantage, so that the power is less than the weight, the distance moved through by the power is therefore greater than the distance moved through by the weight in the same proportion. This is sometimes expressed in popular language in the form; *What is gained in power is lost in speed.*

201. *To find the work done in dragging a body up a rough inclined plane.*

From Art. 187, Case II., we know that the force P_1 which would just move the body up the plane is

$$W(\sin\alpha + \mu\cos\alpha).$$

Hence the work done in dragging it from A to C

$= P_1 \times AC$

$= W(\sin\alpha + \mu\cos\alpha) \,.\, AC$

$= W \,.\, AC\sin\alpha + \mu W \,.\, AC\cos\alpha$

$= W \,.\, BC + \mu W \,.\, AB$

= work done in dragging the body through the same vertical height without the intervention of the plane

+ the work done in dragging it along a horizontal distance equal to the base of the inclined plane and of the same roughness as the plane.

202. From the preceding article we see that, if our inclined plane be rough, the work done by the power is more than the work done against the weight. This is again true for any machine; the principle may be expressed thus,

In any machine, the work done by the power is equal to the work done against the weight, together with the work done against the frictional resistances of the machine, and the work done against the weights of the component parts of the machine.

The ratio of the work done on the weight to the work done by the power is, for any machine, called the modulus or efficiency of the machine. We can never get rid entirely of frictional resistances, or make our machine without weight, so that some work must always be lost through these two causes. Hence the modulus of the machine can never be so great as unity. The more nearly the modulus approaches to unity, the better is the machine.

203. The student can verify the truth of the principle, enunciated in Art. 200, for all machines; we shall consider only a few cases.

First system of pulleys (Art. 144).

Neglecting the weights of the pulleys we have, if there be four pulleys,

$$P = \frac{1}{2^4} W.$$

If the weight W be raised through a distance x, the pulley A_2 would, if the distance A_1A_2 remained unchanged, rise a distance x; but, at the same time, the length of the string joining A_1 to the beam is shortened by x, and a portion x of the string therefore slips round A_1; hence, altogether, the pulley A_2 rises through a distance $2x$.

Similarly, the pulley A_3 rises a distance $4x$, and the pulley A_4 a distance $8x$.

Since A_4 rises a distance $8x$, the strings joining it to the beam and to the point at which P is applied both shorten by $8x$.

Hence, since the slack string runs round the pulley A_4, the point of application of P rises through $16x$.

$$\text{Hence } \frac{\text{work done by the power}}{\text{work done against the weight}} = \frac{P \, . \, 16x}{W \, . \, x}$$

$$= \frac{\frac{1}{2^4} W \, . \, 16x}{W \, . \, x} = \frac{W \, . \, x}{W \, . \, x} = 1.$$

Hence the principle is verified.

Taking the weights of the pulleys into account, and taking the case of four pulleys, we have, by Art. 144, Case II.,

$$P=\frac{W}{2^4}+\frac{w_1}{2^4}+\frac{w_2}{2^3}+\frac{w_3}{2^2}+\frac{w_4}{2}.$$

As before, if A_1 ascend a distance x, the other pulleys ascend distances $2x$, $4x$, and $8x$, respectively. Hence the work done on the weight and the weights of the pulleys

$$= W.x+w_1.x+w_2.2x+w_3.4x+w_4.8x$$

$$=16x\left[\frac{W}{2^4}+\frac{w_1}{2^4}+\frac{w_2}{2^3}+\frac{w_3}{2^2}+\frac{w_4}{2}\right]$$

$$=16x\times P=\text{work done by the power.}$$

A similar method of proof would apply, whatever be the number of pulleys.

204. *Third system of pulleys* (Art. 146).

Suppose the weight W to ascend through a space x. The string joining B to the bar shortens by x, and hence the pulley A_3 descends a distance x. Since the pulley A_3 descends x and the bar rises x, the string joining A_3 to the bar shortens by $2x$, and this portion slides over A_3; hence the pulley A_2 descends a distance equal to $2x$ together with the distance through which A_3 descends, *i.e.*, A_2 descends a distance $2x+x$, or $3x$. Hence the string A_2F shortens by $4x$, which slips over the pulley A_2, so that the pulley A_1 descends a distance $4x$ together with the distance through which A_2 descends, *i.e.*, $4x+3x$, or $7x$. Hence the string A_1G shortens by $8x$, and A_1 itself descends $7x$, so that the point of application of P descends $15x$.

Neglecting the weight of the pulleys, the work done by P therefore

$$=15x.P=x(2^4-1)P=x.W \text{ by equation (1), Art. 146,}$$

$$=\text{work done on the weight } W.$$

Taking the weights of the pulleys into account, the work done by the power and the weights of the pulleys [which in this case assist the power]

$$\begin{aligned}&= P\,.\,15x + w_1\,.\,7x + w_2\,.\,3x + w_3\,.\,x\\ &= x\left[P\left(2^4-1\right) + w_1\left(2^3-1\right) + w_2\left(2^2-1\right) + w_3\right]\\ &= x\,.\,W \text{ by equation (3), Art. 146,}\\ &= \text{work done on the weight } W.\end{aligned}$$

205. *Smooth inclined plane* (Art. 151, Case III.).

Let the body move a distance x along the plane; the distance through which the point of application of P moves, measured along its direction of application, is clearly $x\cos\theta$; also the vertical distance through which the weight moves is $x\sin\alpha$.

Hence the work done by the power is $P\,.\,x\cos\theta$, and that done against the weight is $W\,.\,x\sin\alpha$. These are equal by the relation proved in Art. 151.

206. *Wheel and axle* (Art. 153).

Let the machine turn through an angle θ. A portion of string whose length is $b\,.\,\theta$ becomes unwound from the wheel, and hence P descends through this distance. At the same time a portion equal to $a\,.\,\theta$ becomes wound upon the axle, so that W rises through this distance. The work done by P is therefore $P \times b\theta$ and that done against W is $W \times a\theta$.

These are equal by the relation of Art. 154.

207. *The Smooth Screw* (Art. 172).

For each revolution made by the power arm the screw rises through a distance equal to the distance between two consecutive threads.

Hence, during each revolution, the work done by the power is

$P \times$ circumference of the circle described by the end of the power-arm, and that done against the weight is

$W \times$ distance between two consecutive threads.

These are equal by the relation proved in Art. 172.

208. **Theorem.** *To shew that the work done in raising a number of particles from one position to another is Wh, where W is the total weight of the particles, and h is the distance through which the centre of gravity of the particles has been raised.*

Let $w_1, w_2, w_3, \ldots w_n$ be the weights of the particles; in the initial position let $x_1, x_2, x_3, \ldots x_n$ be their heights above a horizontal plane, and $\bar{x}$ that of their centre of gravity, so that, as in Art. 111, we have

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \ldots + w_nx_n}{w_1 + w_2 + \ldots + w_n} \ldots\ldots\ldots\ldots\ldots(1).$$

In the final position let $x_1', x_2', \ldots x_n'$ be the heights of the different particles, and $\bar{x}'$ the height of the new centre of gravity, so that

$$\bar{x}' = \frac{w_1x_1' + w_2x_2' + \ldots w_nx_n'}{w_1 + w_2 + \ldots w_n} \ldots\ldots\ldots\ldots\ldots(2).$$

But, since $w_1 + w_2 + \ldots = W$, equations (1) and (2) give

$$w_1x_1 + w_2x_2 + \ldots = W \cdot \bar{x},$$

and $$w_1x_1' + w_2x_2' + \ldots = W \cdot \bar{x}'.$$

By subtraction we have

$$w_1(x_1' - x_1) + w_2(x_2' - x_2) + \ldots = W(\bar{x}' - \bar{x}).$$

But the left-hand member of this equation gives the total work done in raising the different particles of the system from their initial position to their final position; also the right-hand side

$= W \times$ height through which the centre of gravity has been raised

$= W . h.$

Hence the proposition is proved.

209. **Power. Def.** *The power of an agent is the amount of work that would be done by the agent if working uniformly for the unit of time.*

The unit of power used by engineers is called a **Horse-Power.** An agent is said to be working with one horse-power when it performs 33,000 foot-pounds in a minute, *i.e.*, when it would raise 33,000 lbs. through a foot in a minute, or when it would raise 330 lbs. through 100 feet in a minute.

This estimate of the power of a horse was made by Watt, but is rather above the capacity of ordinary horses. The word Horse-power is usually abbreviated into H. P.

210. **Ex.** *A well, of which the section is a square whose side is 4 feet, and whose depth is 300 feet, is full of water; find the work done, in foot-pounds, in pumping the water to the level of the top of the well.*

Find also the H. P. of the engine which would just accomplish this work in one hour.

[N.B. A cubic foot of water weighs 1000 ounces.]

Initially the height of the centre of gravity of the water above the bottom of the well was 150 feet and finally it is 300 feet, so that the height through which the centre of gravity has been raised is 150 feet.

The volume of the water $= 4 \times 4 \times 300$ cubic feet.

$\therefore$ its weight $= 4 \times 4 \times 300 \times \frac{1000}{16}$ lbs. $= 300,000$ lbs.

Hence the work done $= 300,000 \times 150$ ft.-lbs. $= 45,000,000$ ft.-lbs.

Let x be the required H. P. Then the work done by the engine in one hour $= x \times 60 \times 33,000.$

Hence we have $x \times 60 \times 33,000 = 45,000,000;$

$$\therefore x = 22\tfrac{8}{11}.$$

EXAMPLES. XXXIII.

1. How much work is done in drawing a load of 6 cwt. up a rough inclined plane, whose height is 3 feet and base 20 feet, the coefficient of friction being $\frac{2}{15}$?

2. A weight of 10 tons is dragged in half an hour through a length of 330 feet up an inclined plane, of inclination 30°, the coefficient of friction being $\frac{1}{\sqrt{3}}$; find the work expended and the H. P. of the engine which could do the work.

3. How many cubic feet of water will an engine of 100 H. P. raise in one hour from a depth of 150 feet?

4. A tank, 24 feet long, 12 feet broad, and 16 feet deep, is filled by water from a well the surface of which is always 80 feet below the top of the tank; find the work done in filling the tank, and the H. P. of an engine, whose modulus is ·5, that will fill the tank in 4 hours.

5. A chain, whose mass is 8 lbs. per foot, is wound up from a shaft by the expenditure of four millions units of work; find the length of the chain.

6. In how many hours would an engine of 18 H. P. empty a vertical shaft full of water if the diameter of the shaft be 9 feet, and the depth 420 feet?

7. Find the H. P. of an engine that would empty a cylindrical shaft full of water in 32 hours, if the diameter of the shaft be 8 feet and its depth 600 feet.

8. A tower is to be built of brickwork, the base being a rectangle 22 ft. by 9 ft., the height of the tower 66 feet, and the walls two feet thick; find the number of hours in which an engine of 3 H. P. would raise the bricks from the ground, the weight of a cubic foot of brickwork being 112 lbs.

9. At the bottom of a coal mine, 275 feet deep, there is an iron cage containing coal weighing 14 cwt., the cage itself weighing 4 cwt. 109 lbs., and the wire rope that raises it 6 lbs. per yard. Find the work done when the load has been lifted to the surface, and the H. P. of the engine that can do this work in 40 seconds.

10. The diameter of the circular piston of a steam engine is 60 inches and it makes 11 strokes per minute, the length of each stroke being 8 feet, the mean pressure per square inch on the piston being 15 lbs., and the modulus of the engine ·65. Find the number of cubic feet of water that it will raise per hour from a well whose depth is 300 feet, on the supposition that no work is wasted.

11. In a wheel and axle, if the radius of the wheel be six times that of the axle, and if by means of a power equal to 5 lbs. wt. a body be lifted through 50 feet, find the amount of work expended.

12. A capstan, of diameter 20 inches, is worked by means of a lever, which measures 5 feet from the axis of the capstan. Find the work done in drawing up by a rope a body, of weight one ton, over 35 feet of the surface of a smooth plane inclined to the horizon at an angle $\cos^{-1}\frac{4}{5}$. Find also the force applied to the end of the lever, and the distance through which the point of application moves.

13. A block, of weight one ton, is in the form of a rectangular parallelopiped, 8 feet high, standing on a square base whose side is 6 feet. It is placed on a rough board with the sides of its base parallel to the length and breadth of the board, and the centre of the base is distant 6 feet from one extremity of the board. The board is now tilted round this extremity until the board topples without sliding; find the work done.

14. If the axis of a screw be vertical and the distance between the threads 2 inches, and a door, of weight 100 lbs., be attached to the screw as to a hinge, find the work done in turning the door through a right angle.

CHAPTER XIV.

MISCELLANEOUS.

211. **Bodies connected by smooth hinges.** When two bodies are hinged together, it usually happens that, either a rounded end of one body fits loosely into a prepared hollow in the other body, or that a round pin, or other separate fastening, passes through a hole in each body.

In either case, if the bodies be smooth, the action on each body at the hinge consists of a single force. Also the action of the hinge on the one body is equal and opposite to the action of the hinge on the other body; for forces, equal and opposite to these actions, keep the pin, or fastening, in equilibrium, since its weight is negligible.

In solving questions concerning hinges, the direction and magnitude of the action at the hinge are usually both unknown.

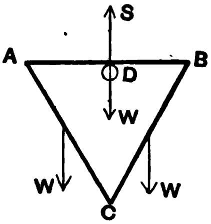
Hence it is generally most convenient to assume the action of the hinge on one body to consist of two unknown components at right angles to one another; the action of the hinge on the other body will then consist of components equal and opposite to these.

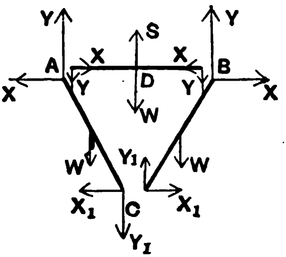
The forces acting on each body, together with the actions of the hinge on it, are in equilibrium, and the general conditions of equilibrium of Art. 83 will now apply.

In order to avoid mistakes as to the components of the reaction acting on each body, it is convenient, as in the second figure of the following example, not to produce the beams to meet but to leave a space between them.

212. **Ex.** *Three equal uniform rods, each of weight W, are smoothly jointed so as to form an equilateral triangle. If the system be supported at the middle point of one of the rods, shew that the action at the lowest angle is* $\frac{W}{2\sqrt{3}}$, *and that at each of the others is* $W\sqrt{\frac{13}{12}}$.

Let ABC be the triangle formed by the rods, and D the middle point of the side AB at which the system is supported.





Let the action of the hinge at A on the rod AB consist of two components, respectively equal to Y and X, acting in vertical and horizontal directions; hence the action of the hinge on AC consists of components equal and opposite to these.

Since the whole system is symmetrical about the vertical line through D, the action at B will consist of components, also equal to Y and X, as in the figure.

Let the action of the hinge C on CB consist of Y_1 vertically upwards, and X_1 horizontally to the right, so that the action of the same hinge on CA consists of two components opposite to these, as in the figure.

For AB, resolving vertically, we have

$$S=W+2Y \quad\text{.............................}(1),$$

where S is the vertical reaction of the peg at D.

For CB, resolving horizontally and vertically, and taking moments about C, we have

$$X+X_1=0 \quad\text{................................}(2),$$

$$W=Y+Y_1 \quad\text{...............................}(3),$$

$$W \,.\, a\cos 60^\circ + X \,.\, 2a\sin 60^\circ = Y \,.\, 2a\cos 60^\circ \quad\text{.........}(4).$$

For CA, by resolving vertically, we have

$$W=Y-Y_1 \quad\text{................................}(5).$$

From equations (3) and (5) we have

$$Y_1=0, \text{ and } Y=W.$$

Hence equation (4) is

$$X=\tfrac{1}{2}W\cot 60^\circ=\frac{W}{2\sqrt{3}}.$$

$\therefore$ from (2) $\qquad X_1=-\dfrac{W}{2\sqrt{3}}.$

Also (1) gives $\qquad S=3W.$

Hence the action of the hinge at B consists of a force $\sqrt{X^2+Y^2}$ $\left(\text{i.e. } W\sqrt{\frac{13}{12}}\right)$, acting at an angle $\tan^{-1}\frac{Y}{X}$ (*i.e.* $\tan^{-1} 2\sqrt{3}$), to the horizon; also the action of the hinge at C consists of a horizontal force equal to $\dfrac{W}{\sqrt{12}}$.

A priori reasoning would have shewn us that the action of the hinge at C must be horizontal; for the whole system is symmetrical about the line CD, and, unless the component Y_1 vanished, the reaction at C would not satisfy the condition of symmetry.

EXAMPLES. XXXIV.

1. Two equal uniform beams, AB and BC, are freely jointed at B and A is fixed to a hinge at a point in a wall about which AB can turn freely in a vertical plane. At what point in BC must a vertical force be applied to keep the two beams in one horizontal line, and what is the magnitude of the force?

2. Two uniform beams, AC and CB, are smoothly hinged together at C, and have their ends attached at two points, A and B, in the same horizontal line. If they be made of the same material and be of total weight 60 lbs., and if each is inclined at an angle of 60° to the horizon, shew that the action of the hinge at the point C is a horizontal form of $5\sqrt{3}$ lbs. weight.

3. A pair of compasses, each of whose legs is a uniform bar of weight W, is supported, hinge downwards, by two smooth pegs placed at the middle points of the legs in the same horizontal line, the legs being kept apart at an angle 2α with one another by a weightless rod joining their extremities; shew that the thrust in this rod and that the action at the hinge are each $\frac{1}{2}W \cot \alpha$.

4. Two uniform rods, AB and AC, each of weight w, are smoothly jointed at A and placed in a vertical plane with the ends B and C resting on a smooth table. Equilibrium is preserved by a string which attaches C to the middle point of AB. Shew that the tension of the string and the reaction of the rods at A are both equal to

$$\frac{W}{4} \operatorname{cosec} \alpha \sqrt{1 + 8 \cos^2 \alpha},$$

and that both are inclined at an angle $\tan^{-1} (\frac{1}{3} \tan \alpha)$ to the horizon, where α is the inclination of either rod to the horizon.

5. Two equal beams, AC and BC, freely jointed together at C, stand with their ends, A and B, in contact with a rough horizontal plane, and with the plane ACB vertical. If the coefficient of friction be $\frac{1}{2}$, shew that the angle ACB cannot be greater than a right angle, and find the thrust at C in any position of equilibrium.

6. Three uniform heavy rods, AB, BC, and CA, of lengths 5, 4, and 3 feet respectively, are hinged together at their extremities to form a triangle. Shew that the whole will balance, with AB horizontal, about a fulcrum which is distant $1\frac{1}{5}$ of an inch from the middle point towards A.

Prove also that the vertical components of the actions at the hinges A and B, when the rod is balanced, are $\frac{187}{600} W$ and $\frac{163}{600} W$ respectively, where W is the total weight of the rods.

7. Two equal rods, AB and BC, are jointed at B, and have their middle points connected by an inelastic string of such a length that, when it is straightened, the angle ACB is a right angle; if the system be suspended by a string attached to the point A, shew that the inclination of AB to the vertical will be $\tan^{-1} \frac{1}{3}$, and find the tension of the string and the action at the hinge.

8. Two equal bars, AB and BC, each 1 foot long and each of weight W, are jointed together at B and suspended by strings OA, OB, and OC, each 1 foot long, from a fixed peg O; find the tensions of the three strings and the magnitude of the action at the hinge.

9. Three uniform beams AB, BC, and CD, of the same thickness, and of lengths l, $2l$, and l respectively, are connected by smooth hinges at B and C, and rest on a perfectly smooth sphere, whose radius is $2l$, so that the middle point of BC and the extremities, A and D, are in contact with the sphere; shew that the pressure at the middle point of BC is $\frac{21}{100}$ of the weight of the beams.

10. Three rods AB, BC, and CD, whose weights are proportional to their lengths a, b, and c, are jointed at B and C and are in a horizontal position resting on two pegs P and Q; find the actions at the joints B and C, and shew that the distance between the pegs must be

$$\frac{a^2}{2a+b}+\frac{c^2}{2c+b}+b.$$

11. AB and AC are similar uniform rods, of length a, smoothly jointed at A. BD is a weightless bar, of length b, smoothly jointed at B, and fastened at D to a smooth ring sliding on AC. The system is hung on a small smooth pin at A. Shew that the rod AC makes with the vertical an angle

$$\tan^{-1}\frac{b}{a+\sqrt{a^2-b^2}}.$$

12. A square figure $ABCD$ is formed by four equal rods jointed together, and the system is suspended from the joint A, and kept in the form of a square by a string connecting A and C; shew that the tension of the string is half the weight of the four rods, and find the direction and magnitude of the action at either of the joints B or D.

13. Four equal rods are jointed together to form a rhombus, and the opposite joints are joined by strings forming the diagonals. Shew that their tensions are in the same ratio as their lengths.

213. **Funicular Polygon.** If a light string have its ends attached to two fixed points, and if at different points of the string there be attached weights, the figure formed by the string is called a funicular polygon.

Let O and O_1 be the two fixed points at which the ends of the string are tied, and let A_1, A_2, ... A_n, be the points of the string at which are attached bodies, whose weights are w_1, w_2, ... w_n respectively.

Let the lengths of the portions

OA_1, A_1A_2, A_2A_3, ... A_nO_1, be a_1, a_2, a_3, ... a_{n+1}, respectively, and let their inclinations to the horizon be

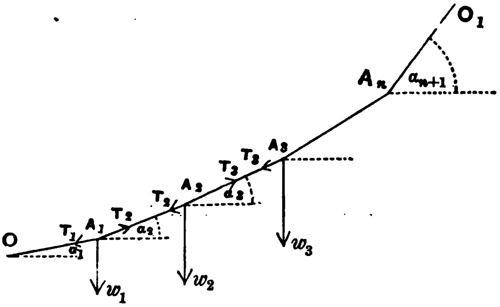
$$\alpha_1,\quad \alpha_2,\quad \dots\ \alpha_{n+1}.$$

Let h and k be respectively the horizontal and vertical distances between the points O and O_1, so that

$$a_1 \cos \alpha_1 + a_2 \cos \alpha_2 + \ldots + a_{n+1} \cos \alpha_{n+1} = h \ldots (1),$$

and

$$a_1 \sin \alpha_1 + a_2 \sin \alpha_2 + \ldots + a_{n+1} \sin \alpha_{n+1} = k \ldots (2).$$



Let $T_1, T_2, \ldots T_{n+1}$ be respectively the tensions of the portions of the string.

Resolving vertically and horizontally for the equilibrium of the different weights in succession, we have

$$T_2 \sin \alpha_2 - T_1 \sin \alpha_1 = w_1, \text{ and } T_2 \cos \alpha_2 - T_1 \cos \alpha_1 = 0;$$

$$T_3 \sin \alpha_3 - T_2 \sin \alpha_2 = w_2, \text{ and } T_3 \cos \alpha_3 - T_2 \cos \alpha_2 = 0;$$

$$T_4 \sin \alpha_4 - T_3 \sin \alpha_3 = w_3, \text{ and } T_4 \cos \alpha_4 - T_3 \cos \alpha_3 = 0;$$

……………………………………………………………

$$T_{n+1} \sin \alpha_{n+1} - T_n \sin \alpha_n = w_n, \text{ and } T_{n+1} \cos \alpha_{n+1} - T_n \cos \alpha_n = 0.$$

These $2n$ equations, together with the equations (1) and (2), are sufficient to determine the $(n+1)$ unknown tensions, and the $(n+1)$ unknown inclinations

$$\alpha_1, \quad \alpha_2, \quad \ldots \quad \alpha_{n+1}.$$

From the right-hand column of equations, we have

$$T_1 \cos \alpha_1 = T_2 \cos \alpha_2 = T_3 \cos \alpha_3 = \ldots = T_{n+1} \cos \alpha_{n+1}$$
$$= K \text{ (say)} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

so that the horizontal component of the tension of the string is constant throughout and is denoted by K.

From (3), substituting for T_1, T_2, ... T_{n+1} in the left-hand column of equations, we have

$$\tan\alpha_2 - \tan\alpha_1 = \frac{w_1}{K},$$

$$\tan\alpha_3 - \tan\alpha_2 = \frac{w_2}{K},$$

$$\tan\alpha_4 - \tan\alpha_3 = \frac{w_3}{K},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tan\alpha_{n+1} - \tan\alpha_n = \frac{w_n}{K}.$$

If the weights be all equal, the right-hand members of this latter column of equations are all equal and it follows that $\tan\alpha_1$, $\tan\alpha_2$, ... $\tan\alpha_{n+1}$, are in arithmetical progression.

Hence when a set of equal weights are attached to different points of a string, as above, the tangents of inclination to the horizon of successive portions of the string form an arithmetical progression whose constant difference is the weight of any attached particle divided by the constant horizontal tension of the strings.

214. If, in the Funicular Polygon, the inclinations of the different portions of string be given, we can easily, by geometric construction, obtain the values of w_1, w_2, ... w_n.

For let C be any point and CD the horizontal line through C. Draw CP_1, CP_2, ... CP_{n+1} parallel to the strings CA_1, A_1A_2, ... A_nO_1, so that the angles P_1CD, P_2CD, ... are respectively α_1, α_2, ...

Draw any vertical line cutting these lines in D, P_1, P_2 ....

Then, by the previous article,

$$\frac{w_1}{K} = \tan\alpha_2 - \tan\alpha_1 = \frac{DP_2}{CD} - \frac{DP_1}{CD} = \frac{P_2P_1}{CD},$$

$$\frac{w_2}{K} = \tan\alpha_3 - \tan\alpha_2 = \frac{DP_3}{CD} - \frac{DP_2}{CD} = \frac{P_3P_2}{CD},$$

and so on.

Hence the quantities K, w_1, w_2, ... w_n are respectively proportional to the lines CD, P_1P_2, P_2P_3, ... P_nP_{n+1}, and hence their ratios are determined.

215. **Tensions of Elastic Strings.** All through this book we have assumed our strings to be inextensible, *i.e.*, that they would bear any tension without altering their length.

In practice, all strings are extensible, although the extensibility is in many cases extremely small, and practically negligible.

When the extensibility of the string cannot be neglected, there is a simple experimental law connecting the tension of the string with the amount of extension of the string. It may be expressed in the form

The tension of an elastic string varies as the extension of the string beyond its natural length.

Suppose a string to be naturally of length one foot; its tension, when the length is 13 inches, will be to its tension, when of length 15 inches, as

$$13-12 \; : \; 15-12, \text{ i.e., as } 1 : 3.$$

The student will observe carefully that the tension of the string is not proportional to its stretched length, but to its extension.

The above law was discovered by Hooke, and enunciated by him in the form *Ut tensio, sic vis.*

From Hooke's law we easily obtain a formula giving us the tension in any case.

Let a be the unstretched length of a string, and T be its tension when it is stretched to be of length x. The extension is now $x-a$, and the law states that

$$T \propto x-a.$$

This is generally expressed in the form

$$T=\lambda\,.\,\frac{x-a}{a},$$

the constant of variation being $\dfrac{\lambda}{a}$.

The quantity λ is a quantity depending on the material of which the string is made, and is called the *Modulus of Elasticity of the String.*

The quantity λ is equal to the force which would stretch the string, if placed on a smooth horizontal table, to twice its natural length; for, when $x=2a$, we have the tension

$$=\lambda\frac{2a-a}{a}=\lambda.$$

Hence the value of λ may be easily determined by finding the force which stretches the string to twice its natural length.

No elastic string will however bear an unlimited stretching; when the string, through being stretched, is on the point of breaking, it is said to have reached its *limit of elasticity*, and its tension then is called the *breaking tension.*

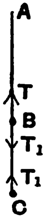
Ex. *ABC is an elastic string, hanging vertically from a fixed point A; at B and C are attached particles, of weights 2W and W respectively. If the modulus of elasticity of the string be 3W, find the ratio of the stretched lengths of the portions of the string to their unstretched lengths.*

Let c and c_1 be the unstretched lengths of AB and BC, and x and y their stretched lengths.

Let T and T_1 be their tensions, so that

$$T=\lambda\frac{x-c}{c}=3W\frac{x-c}{c},$$

and $$T_1=\lambda\frac{y-c_1}{c_1}=3W\frac{y-c_1}{c_1}.$$



From the equilibrium of B and C, we have

$$T-T_1=2W, \text{ and } T_1=W.$$

Hence $$T=3W.$$

$$\therefore\ 3W\frac{x-c}{c}=3W, \text{ and } 3W\frac{y-c_1}{c_1}=W.$$

$$\therefore\ x=2c, \text{ and } y=\tfrac{4}{3}c_1,$$

so that the stretched lengths are respectively twice and four-thirds of the natural lengths.

EXAMPLES. XXXV.

1. ABC is an elastic string, whose modulus of elasticity is $4W$, which is tied to a fixed point at A. At B and C are attached weights each equal to W, the unstretched lengths of AB and BC being each equal to c. Shew that, if the string and bodies take up a vertical position of equilibrium, the stretched lengths of AB and BC are $\frac{3}{2}c$ and $\frac{5}{4}c$ respectively.

2. An elastic string has its ends attached to two points in the same horizontal plane, and initially it is just tight and unstretched; a particle, of weight W, is tied to the middle point of the string; if the modulus of elasticity be $\frac{W}{\sqrt{3}}$, shew that, in the position of equilibrium, the two portions of the string will be inclined at an angle of 60° to one another.

3. In the previous question, if $2a$ be the distance between the two points, $2c$ the unstretched length of the string, and λ the modulus of elasticity, shew that the inclination, θ, of the strings to the vertical is given by

$$\frac{W}{2\lambda}\tan\theta+\sin\theta=\frac{a}{c}.$$

4. Two equal heavy rods, AB and CD, are connected crosswise by two equal elastic strings AD and BC. If the unstretched length of each string be equal to the length of either rod, and the modulus of elasticity to $\frac{1}{\sqrt{3}}$ times the weight of either, shew that, when AB is held horizontally, the system is in equilibrium when the strings are inclined at 30° to the vertical.

5. Four equal jointed rods, each of length a, are hung from an angular point, which is connected by an elastic string with the opposite point. If the rods hang in the form of a square, and if the modulus of elasticity of the string be equal to the weight of a rod, shew that the unstretched length of the string is $\frac{a\sqrt{2}}{3}$.

216. **Examples of graphical solution.** Questions are often propounded to be solved by graphical methods.

Many problems which would be exceedingly difficult or, at any rate, very laborious, to solve by analytical methods are comparatively easy to solve graphically.

These questions are of common occurrence in engineering and other practical work. There is generally little else involved besides the use of the Triangle of Forces and Polygon of Forces.

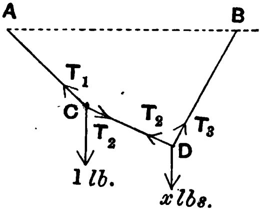
The instruments chiefly used are:—Compasses, Rulers, Scales and Diagonal Scales, and Protractors for measuring angles.

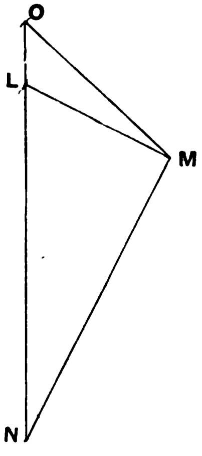
The results obtained are of course not mathematically accurate; but, if the student be careful, and skilful in the use of his instruments, the answer ought to be trustworthy, in general, to the first place of decimals.

In the following worked out examples the figures are reduced from the original drawings; the student is recommended to re-draw them for himself on the scale mentioned in each example.

217. **Ex. 1.** *ACDB is a string whose ends are attached to two points, A and B, which are in a horizontal line and are seven feet apart. The lengths of AC, CD, and DB are* $3\frac{1}{2}$, 3, *and* 4 *feet respectively, and at C is attached a one-pound weight. An unknown weight is attached to D of such a magnitude that, in the position of equilibrium, CDB is a right angle. Find the magnitude of this weight and the tensions of the strings.*

Let T_1, T_2, and T_3 be the required tensions and let x lbs. be the weight at D.





Take a vertical line OL, one inch in length, to represent the weight, one pound, at C. Through O draw OM parallel to AC, and through L draw LM parallel to CD.

By the triangle of forces OM represents T_1, and LM represents T_2.

Produce OL vertically downwards and through M draw MN parallel to BD.

Then, since LM represents T_2, it follows that T_3 is represented by MN, and x by LN.

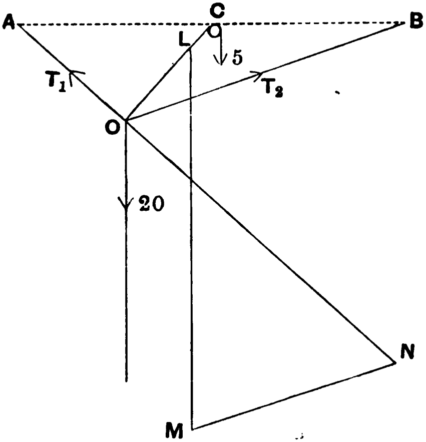
By actual measurement, we have

$$OM = 3{\cdot}05 \text{ ins.}; \quad LM = 2{\cdot}49 \text{ ins.}; \quad MN = 5{\cdot}1 \text{ ins.};$$

and $$NL = 5{\cdot}63 \text{ ins.}$$

Hence the weight at D is 5·63 lbs. and the tensions are respectively 3·05, 2·49, and 5·1 lbs. wt.

Ex. 2. *A and B are two points in a horizontal line at a distance of* 16 *feet apart; AO and OB are two strings of lengths* 6 *and* 12 *feet carrying, at O, a body of weight* 20 *lbs.; a third string, attached to the body at O, passes over a small smooth pulley at the middle point, C, of AB and is attached to a body of weight* 5 *lbs.; find the tensions of the strings AO and OB.*



Let T_1 and T_2 be the required tensions. On OC mark off OL, equal to one inch, to represent the tension, 5 lbs. wt., of the string OC. Draw LM vertical and equal to 4 inches.

Through M draw MN, parallel to OB, to meet AO produced in N.

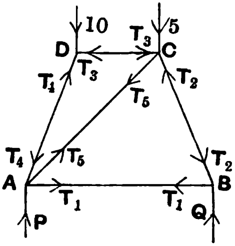
Then, by the Polygon of Forces, the lines ON and NM will represent the tensions T_1 and T_2.

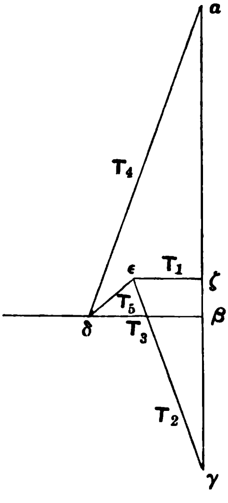
On measurement, ON and NM are found to be respectively 3·9 and 2·45 inches.

Hence $T_1 = 5 \times 3{\cdot}9 = 19{\cdot}5$ lbs. wt.,

and $T_2 = 5 \times 2{\cdot}45 = 12{\cdot}25$ lbs. wt.

Ex. 3. *A framework, $ABCD$, stiffened by a brace AC, is supported in a vertical plane by supports at A and B, so that AB is horizontal; the lengths of AB, BC, CD, DA are 4, 3, 2, and 3 ft. respectively; also AB and CD are parallel, and AD and BC are equally inclined to AB. If weights of 5 and 10 cwt. respectively be placed at C and D, find the reactions of the supports at A and B, and the forces exerted by the different portions of the framework.*





Let the forces in the sides be as marked in the figure and let P and Q be the reactions at A and B.

Draw a vertical line $\alpha\beta$, 5 inches in length, to represent the weight 10 cwt. at D; also draw $\alpha\delta$ parallel to AD and $\beta\delta$ parallel to CD. Then $\alpha\beta\delta$ is the Triangle of Forces for the joint D, and the forces at D must be in the directions marked.

Note that the force at C in the bar DC must be along DC, and that at A along DA.

[This is an important general principle; for any bar, which undergoes stress, is either resisting a tendency to compress it, or a tendency to stretch it.

In the first case, the action at each end is from its centre towards its ends; in the second case it is towards its centre.

In *either* case the actions at the two ends of the rod are equal and opposite.]

Draw $\beta\gamma$ vertical and equal to $2\frac{1}{2}$ ins. to represent the weight at C. Draw $\gamma\epsilon$ parallel to BC and $\delta\epsilon$ parallel to AC. Then $\delta\beta\gamma\epsilon\delta$ is the Polygon of Forces for the joint C, so that the actions at C are as marked.

Draw $\epsilon\zeta$ horizontal to meet $\alpha\gamma$ in β.

Then $\epsilon\gamma\zeta$ is the Triangle of Forces for B, so that the reaction Q is represented by $\gamma\zeta$, and T_1 by $\zeta\epsilon$.

Finally, for the joint A, we have the polygon $\delta\epsilon\zeta\alpha\delta$, so that P is represented by $\alpha\zeta$.

On measuring, we have, in inches,

$$\epsilon\zeta = 1{\cdot}10;\quad \gamma\epsilon = 3{\cdot}31;\quad \delta\beta = 1{\cdot}77;\quad \delta\alpha = 5{\cdot}30;\quad \delta\epsilon = {\cdot}91;$$
$$\gamma\zeta = 3{\cdot}125;\quad \beta\alpha = 4{\cdot}375.$$

Hence, since one inch represents 2 cwt., we have, in cwts.,

$$T_1 = 2{\cdot}20;\quad T_2 = 6{\cdot}62;\quad T_3 = 3{\cdot}54;\quad T_4 = 10{\cdot}6;\quad T_5 = 1{\cdot}82;$$
$$Q = 6{\cdot}25;\quad P = 8{\cdot}75.$$

It will be noted that the bars AB and AC are in a state of tension, whilst the rest of the framework is in a state of compression.

EXAMPLES. XXXVI.

The following are to be solved by graphic methods.

1. A boat is towed along a river by means of two ropes, attached to the same point, which are pulled by two men who keep at opposite points of the bank 50 feet apart; one rope is 30 feet long and is pulled with a force equal to the weight of 35 lbs., and the other rope is 45 feet long; the boat is in this way made to move uniformly in a straight line; find the resistance offered to the boat by the stream and the tension of the second rope.

2. The jib of a crane is 10 feet long, and the tie-rod is horizontal and attached to a point 6 feet vertically above the foot of the jib; find the tension of the tie-rod, and the thrust on the jib, when the crane supports a mass of 1 ton.

3. A uniform triangular lamina ABC, of 30 lbs. weight, can turn in a vertical plane about a hinge at B; it is supported with the side AB horizontal by a peg placed at the middle point of BC. If the sides AB, BC, and CA be respectively 6, 5, and 4 feet in length, find the pressure on the prop and the strain on the hinge.

4. A ladder, 30 feet long, rests with one end against a smooth wall and the other against the rough ground, the distance of its foot from the wall being 10 feet; find the resultant force exerted by the ground on the foot of the ladder if the weight of the ladder be 150 lbs. (1) when there is no extra weight on the ladder, (2) when 1 cwt. is placed $\frac{3}{4}$ of the way up.

5. It is found by experiment that a force equal to the weight of 10 lbs. acting along the plane is required to make a mass of 10 lbs. begin to move up a plane inclined at 45° to the horizon; find the coefficient of friction between the mass and the plane.

6. Three forces equal respectively to the weights of 5·05 lbs., 4·24 lbs., and 3·85 lbs. act at three given points of a flat disc resting on a smooth table. Place the forces, by geometric construction, so as to keep the disc in equilibrium, and measure the number of degrees in each of the angles which they make with one another.

7. A uniform rectangular block, of which $ABCD$ is the symmetrical section through its centre of gravity, rests with CD in contact with a rough horizontal plane ($\mu=\frac{1}{2}$); the weight of the block is 40 lbs. and a force equal to 10 lbs. wt. acts at D in the direction CD; if the lengths of BC, CD be respectively 3 and 5 feet, find the value of the least force which, applied at the middle point of CB parallel to the diagonal DB, would move the block.

8. A body, of weight 100 lbs., rests on a rough plane whose slope is 1 in 3, the coefficient of friction being $\frac{1}{2}$; find the magnitude of the force which, acting at an angle of 40° with the plane, is on the point of dragging the body up the plane. Find also the force which, acting at an angle of 40° with the plane, is on the point of dragging the body down the plane.

9. A and B are two fixed points, the horizontal and vertical distances between them being 4 feet and 1 foot respectively; AC and BC are strings of length 5 and 3 feet respectively, and at C is tied a body of weight 1 cwt.; find the tensions of the strings.

10. $ABCD$ is a light string attached to two points, A and D, in the same horizontal line, and at the points B and C are attached weights. In the position of equilibrium the distances of the points B and C below the line AD are respectively 4 and 6 feet. If the lengths of AB and CD be respectively 6 and 8 feet and the distance AD be 14 feet, find the weight at C, the magnitude of the weight at B being 4 lbs.

11. AC, CB are two equal beams inclined to one another at an angle of 40°, the ends A, B resting on the ground, which is rough enough to prevent any slipping, and the plane ACB being inclined at an angle of 70° to the ground. At C is attached a body of weight 10 cwt., and the system is supported by a rope, attached to C, which is in the vertical plane passing through C and the middle point of AB. If the rope be attached to the ground and be inclined at an angle of 50° to the ground, find the tension of the rope and the action along the beams.

12. A beam, AB, of weight 140 lbs., rests with one end A on a rough horizontal plane, the other end, B, being supported by a cord, passing over a smooth pulley at C, whose horizontal and vertical distances from A are respectively 15 and 20 feet. If the length of the beam be 15 feet, and it be on the point of slipping when the end B is at a height of 9 feet above the horizontal plane, find the magnitudes of the coefficient of friction, the tension of the cord, and the total pressure at A.

13. A framework ABC is supported in a vertical position and with AB horizontal by supports at A and B; if the lengths AB, BC, and CA be 10, 7, and 9 feet respectively, and a weight of 10 cwt. be placed at C, find the reactions at A and B and the forces exerted by the different portions of the framework.

14. A triangular framework ABC, formed of three bars jointed at its angular points, is in equilibrium under the action of three forces P, Q, and R acting outwards at its angular points, the line of action of each being the line joining its point of application to the middle of the opposite bar. If the sides BC, CA, and AB be 9 ft., 8 ft. and 7 ft. in length respectively, and if the force P be equal to 50 lbs. wt., find the values of Q and R, and the forces acting along the bars of the framework.

15. A triangular framework formed of three bars jointed at the angular points is in equilibrium under the action of three given forces acting at its angular points. Draw a diagram to represent to scale the given forces and the stresses in the bars.

MISCELLANEOUS EXAMPLES.

1. If O be the centre of the circle circumscribing the triangle ABC, and if forces act along OA, OB, and OC, respectively proportional to BC, CA, and AB, shew that their resultant passes through the centre of the inscribed circle.

2. Three forces act along the sides of a triangle ABC, taken in order, and their resultant passes through the orthocentre and the centre of gravity of the triangle; shew that the forces are in the ratio of

$$\sin 2A \sin (B - C) \ : \ \sin 2B \sin (C - A) \ : \ \sin 2C \sin (A - B).$$

Shew also that their resultant acts along the line joining the centres of the inscribed and circumscribing circles, if the forces be in the ratio

$$\cos B - \cos C \ : \ \cos C - \cos A \ : \ \cos A - \cos B.$$

3. Three forces PA, PB, and PC, diverge from the point P and three others AQ, BQ, and CQ converge to a point Q. Shew that the resultant of the six is represented in magnitude and direction by $3PQ$ and that it passes through the centre of gravity of the triangle ABC.

4. T is the orthocentre, and O the circumcentre of a triangle ABC; shew that the three forces AT, BT, and CT have as resultant the force represented by twice OT.

5. Find the centre of gravity of three particles placed at the centres of the escribed circles of a triangle, if they be inversely proportional to the radii of these circles.

6. $ABCD$ is a rectangle; find a point P in AD such that, when the triangle PDC is taken away, the remaining trapezoid $ABCP$ may, when suspended from P, hang with its sides AP and BC horizontal.

7. A triangular lamina ABC, obtuse-angled at C, stands with the side AC in contact with a table. Shew that the least weight, which suspended from B will overturn the triangle, is

$$\tfrac{1}{3}W \frac{a^2 + 3b^2 - c^2}{c^2 - a^2 - b^2},$$

where W is the weight of the triangle.

Interpret the above if $c^2 > a^2 + 3b^2$.

8. A pack of cards is laid on a table, and each card projects in the direction of the length of the pack beyond the one below it; if each project as far as possible, shew that the distances between the extremities of successive cards will form a harmonical progression.

9. If aA, bB, $cC\ldots$ represent n forces, whose points of application are $a, b, c, \ldots$ and whose extremities are $A, B, C, \ldots$, shew that their resultant is given in magnitude and direction by $n \,.\, gG$, where g is the centre of inertia of n equal particles $a, b, c, \ldots$, and G the centre of inertia of n equal particles $A, B, C, \ldots$.

What follows if g coincide with G?

10. From a body, of weight W, a portion, of weight w, is cut out and moved through a distance x; shew that the centre of gravity of the whole moves through a distance $\frac{xw}{W}$ in the same direction.

Shew also that the line joining the two positions of the centre of gravity of the whole body is parallel to the line joining the two positions of the centre of gravity of the part moved.

11. Two uniform rods, AB and AC, of the same material are rigidly connected at A, the angle BAC being 60°, and the length of AB being double that of AC. If G be the centre of inertia of the rods, shew that $BG = AC\sqrt{\frac{19}{12}}$, and, if the system be suspended freely from the end B of the rod AB, shew that the action at A consists of a vertical force equal to one-third of the weight, W, of the system, and a couple whose moment is

$$\tfrac{3}{8}W\frac{AC}{\sqrt{19}}.$$

12. If the hinges of a gate be 4 feet apart and the gate be 10 feet wide and weigh 500 lbs., shew that, on the assumption that all the weight is borne by the lower hinge, the stress on the upper hinge must be 625 lbs. wt.

13. A step-ladder in the form of the letter A, with each of its legs inclined at an angle α to the vertical, is placed on a horizontal floor, and is held up by a cord connecting the middle points of its legs, there being no friction anywhere; shew that, when a weight W is placed on the top of the ladder, the tension of this cord is increased by $W\tan\alpha$.

Shew also that, when the weight W is placed on one of the steps at a height from the floor equal to $\frac{1}{n}$ of the height of the ladder, the increase of tension is $\frac{1}{n}W\tan\alpha$.

14. A cylinder, of radius r, whose axis is fixed horizontally, touches a vertical wall along a generating line. A flat beam of uniform material, of length $2l$ and weight W, rests with its extremities in contact with the wall and the cylinder, making an angle of 45° with the vertical. Shew that, in the absence of friction, $\frac{l}{r} = \frac{\sqrt{5}-1}{\sqrt{10}}$; that the pressure on the wall is $\frac{1}{2}W$, and that the pressure on the cylinder is $\frac{1}{2}\sqrt{5}W$.

15. A uniform rod, of length $32a$, rests partly within and partly without a smooth cylindrical cup of radius a. Shew that in the position of equilibrium the rod makes an angle of 60° with the horizon, and prove also that the cylinder will topple over unless its weight be at least six times that of the rod.

16. A tipping basin, whose interior surface is spherical, is free to turn round an axis at a distance c below the centre of the sphere and at a distance a above the centre of gravity of the basin, and a heavy ball is laid at the bottom of the basin; shew that it will tip over if the weight of the ball exceed the fraction $\frac{a}{c}$ of the weight of the basin.

17. A thin hemispherical shell, closed by a plane base, is filled with water and, when suspended from a point on the rim of the base, it hangs with the base inclined at an angle α to the vertical. Shew that the ratio of the weight of the water to that of the shell is $\tan\alpha - \frac{1}{8} : \frac{3}{8} - \tan\alpha$.

18. A hollow cylinder, composed of thin metal open at both ends, of radius a and height $4a$, is placed on a smooth horizontal plane. Inside it are placed two smooth spheres, of radius r, one above the other, $2r$ being $> a$ and $< 2a$. If W be the weight of the cylinder and W' the weight of one of the spheres, shew that the cylinder will just stand upright, without tumbling over, if

$$W = 2W' \frac{a-r}{a}.$$

19. An isosceles triangular lamina, with its plane vertical, rests, vertex downwards, between two smooth pegs in the same horizontal line; shew that there will be equilibrium if the base make an angle $\sin^{-1}(\cos^2\alpha)$ with the vertical, 2α being the vertical angle of the lamina and the length of the base being three times the distance between the pegs.

20. A prism, whose cross section is an equilateral triangle, rests with two edges horizontal on smooth planes inclined at angles α and β to the horizon. If θ be the angle that the plane through these edges makes with the vertical, shew that

$$\tan\theta = \frac{2\sqrt{3}\sin\alpha\sin\beta + \sin(\alpha+\beta)}{\sqrt{3}\sin(\alpha \sim \beta)}.$$

21. A thin board in the form of an equilateral triangle, of weight 1 lb., has one-quarter of its base resting on the end of a horizontal table, and is kept from falling over by a string attached to its vertex and to a point on the table in the same vertical plane as the triangle. If the length of the string be double the height of the vertex of the triangle above the base, find its tension.

22. A cone, of height h and semi-vertical angle α, is placed with its base against a smooth vertical wall and is supported by a string attached to its vertex and to a point in the wall; shew that the greatest possible length of the string is $h\sqrt{1 + \frac{16}{9}\tan^2\alpha}$.

23. The altitude of a cone is h and the radius of its base is r; a string is fastened to the vertex and to a point on the circumference of the circular base, and is then put over a smooth peg; shew that, if the cone rest with its axis horizontal, the length of the string must be $\sqrt{h^2+4r^2}$.

24. Three equal smooth spheres on a smooth horizontal plane are in contact with one another, and are kept together by an endless string in the plane of their centres, just fitting them; if a fourth equal sphere be placed on them, shew that the tension of the string is to the weight of either sphere as $1 : 3\sqrt{6}$.

25. A smooth rod, of length $2a$, has one end resting on a plane of inclination α to the horizon, and is supported by a horizontal rail which is parallel to the plane and at a distance c from it. Shew that the inclination θ of the rod to the inclined plane is given by the equation $c \sin \alpha = a \sin^2\theta \cos(\theta - \alpha)$.

Shew also that the equilibrium is stable, or unstable, according as the end of the rod rests at a point above, or below, the foot of the perpendicular drawn from the rail upon the plane.

26. A square board is hung flat against a wall, by means of a string fastened to the two extremities of the upper edge and hung round a perfectly smooth rail; when the length of the string is less than the diagonal of the board, shew that there are three positions of equilibrium.

Is the position of symmetry stable or unstable?

27. A hemispherical bowl, of radius r, rests on a smooth horizontal table and partly inside it rests a rod, of length $2l$ and of weight equal to that of the bowl. Shew that the position of equilibrium is given by the equation

$$l \sin(\alpha+\beta) = r \sin \alpha = -2r \cos(\alpha + 2\beta),$$

where α is the inclination of the base of the hemisphere to the horizon, and 2β is the angle subtended at the centre by the part of the rod within the bowl.

28. A uniform rod, of weight W, is suspended horizontally from two nails in a wall by means of two vertical strings, each of length l, attached to its ends. A smooth weightless wedge, of vertical angle 30°, is pressed down with a vertical force $\frac{W}{2}$ between the wall and the rod, without touching the strings, its lower edge being kept horizontal and one face touching the wall. Find the distance through which the rod is thrust from the wall.

29. AB is a smooth plane inclined at an angle α to the horizon, and at A, the lower end, is a hinge about which there works, without friction, a heavy uniform smooth plank AC, of length $2a$. Between the plane and the plank is placed a smooth cylinder, of radius r, which is prevented from sliding down the plane by the pressure of the plank from above. If W be the weight of the plank, W' that of the cylinder, and θ the angle between the plane and the plank, shew that

$$\frac{W'r}{Wa} = \cos(\alpha+\theta)\,\frac{1-\cos\theta}{\sin\alpha}.$$

30. Two equal circular discs—of radius r—with smooth edges, are placed on their flat sides in the corner between two smooth vertical planes inclined at an angle 2α, and touch each other in the line bisecting the angle; prove that the radius of the least disc that can be pressed between them, without causing them to separate, is $r(\sec\alpha - 1)$.

31. A rectangular frame $ABCD$ consists of four freely jointed bars, of negligible weight, the bar AD being fixed in a vertical position. A weight is placed on the upper horizontal bar AB at a given point P and the frame is kept in a rectangular shape by a string AC. Find the tension of the string, and shew that it is unaltered if this weight be placed on the lower bar CD vertically under its former position.

32. A rod MN has its ends in two fixed straight rough grooves OA, OB, in the same vertical plane, which make angles α and β with the horizon; shew that, when the end M is on the point of slipping in the direction AO, the tangent of the angle of inclination of MN to the horizon is $\dfrac{\sin(\alpha-\beta-2\epsilon)}{2\sin(\beta+\epsilon)\sin(\alpha-\epsilon)}$.

33. A rod, resting on a rough inclined plane, whose inclination to the horizon is greater than the angle of friction λ, is free to turn about one of its ends, which is attached to the plane; shew that, for equilibrium, the greatest possible inclination of the rod to the line of greatest slope is $\sin^{-1}(\tan\lambda\cot\alpha)$.

34. Two equal uniform rods, of length $2a$, are jointed at one extremity by a hinge, and rest symmetrically upon a rough fixed sphere of radius c. Find the limiting position of equilibrium, and shew that, if the coefficient of friction be $\dfrac{c}{a}$, the limiting inclination of each rod to the vertical is $\tan^{-1}\sqrt[3]{\dfrac{c}{a}}$.

35. A straight rod, of length $2c$, is placed in a horizontal position as high as possible within a hollow rough sphere, of radius a. Shew that the line joining the middle point of the rod to the centre of the sphere makes with the vertical an angle $\tan^{-1}\dfrac{\mu a}{\sqrt{a^2-c^2}}$.

36. A rough rod is fixed in a horizontal position, and a uniform rod, having one end freely jointed to a fixed point, is in equilibrium resting on the fixed rod; if the perpendicular from the fixed point upon the fixed rod be of length b and be inclined to the horizon at an angle α, shew that the portion of the fixed rod upon any point of which the movable rod may rest is of length

$$\frac{2\mu b\cos\alpha}{\sqrt{\sin^2\alpha-\mu^2\cos^2\alpha}},$$

where μ is the coefficient of friction.

37. A glass rod is balanced, partly in and partly out of a cylindrical tumbler, with the lower end resting against the vertical side of the tumbler. If α and β be the greatest and least angles which the rod can make with the vertical, shew that the angle of friction is

$$\tfrac{1}{2}\tan^{-1}\frac{\sin^3\alpha-\sin^3\beta}{\sin^2\alpha\cos\alpha+\sin^2\beta\cos\beta}.$$

38. A rod rests partly within and partly without a box in the shape of a rectangular parallelepiped, and presses with one end against the rough vertical side of the box, and rests in contact with the opposite smooth edge. The weight of the box being four times that of the rod, shew that if the rod be about to slip and the box be about to tumble at the same instant, the angle that the rod makes with the vertical is $\frac{1}{2}\lambda+\frac{1}{2}\cos^{-1}(\frac{1}{3}\cos\lambda)$, where λ is the coefficient of friction.

39. A uniform heavy rod lies on a horizontal table and is pulled perpendicularly to its length by a string attached to any point. About what point will it commence to turn?

Shew also that the ratio of the forces, required to move the rod, when applied at the centre and through the end of the rod perpendicular to the rod, is $\sqrt{2}+1:1$.

40. Two equal heavy particles are attached to a light rod at equal distances c, and two strings are attached to it at equal distances a from the middle point; the rod is then placed on a rough horizontal table, and the strings are pulled in directions perpendicular to the rod and making the same angle θ with the vertical on opposite sides of the rod. Find the least tensions that will turn the rod and shew that, if the coefficient of friction be $\frac{a}{c}$, the tension will be least when θ is $\frac{\pi}{4}$.

41. Two equal similar bodies, A and B, each of weight W, are connected by a light string and rest on a rough horizontal plane, the coefficient of friction being μ. A force P, which is less than $2\mu W$, is applied at A in the direction BA, and its direction is gradually turned through an angle θ in the horizontal plane. Shew that, if P be greater than $\sqrt{2}\mu W$, then both the weights will slip when $\cos\theta = \frac{P}{2\mu W}$, but, if P be less than $\sqrt{2}\mu W$ and be greater than μW, then A alone will slip when $\sin\theta = \frac{\mu W}{P}$.

42. A beam AB lies horizontally upon two others at points A and C; shew that the least horizontal force applied at B in a direction perpendicular to BA, which is able to move the beam, is the less of the two forces $\frac{1}{2}\mu W$ and $\mu W \frac{b-a}{2a-b}$, where AB is $2a$, AC is b, W is the weight of the beam, and μ the coefficient of friction.

43. A rough beam, of length $2a$, AB, is placed horizontally on two equal and equally rough balls, the distance between whose centres is b, touching them in C and D; shew that, if b be not greater than $\frac{4a}{3}$, a position of the beam can be found in which a force P exerted at B perpendicular to the beam will cause it to be on the point of motion both at C and D at the same time.

44. A uniform heavy beam is placed, in a horizontal position, between two unequally rough fixed planes, inclined to the horizon at given angles, in a vertical plane perpendicular to the planes. Find the condition that it may rest there.

45. A uniform rod is in limiting equilibrium, one end resting on a rough horizontal plane and the other on an equally rough plane inclined at an angle α to the horizon. If λ be the angle of friction, and the rod be in a vertical plane, shew that the inclination, θ, of the rod to the horizon is given by

$$\tan\theta = \frac{\sin(\alpha - 2\lambda)}{2\sin\lambda\sin(\alpha - \lambda)}.$$

Find also the normal pressures on the planes.

46. If a pair of compasses rest across a smooth horizontal cylinder of radius c, shew that the frictional couple at the joint to prevent the legs of the compasses from slipping must be

$$W(c\cot\alpha\operatorname{cosec}\alpha - a\sin\alpha),$$

where W is the weight of each leg, 2α the angle between the legs, and a the distance of the centre of gravity of a leg from the joint.

47. The handles of a drawer are equidistant from the sides of the drawer and are distant c from each other; shew that it will be impossible to pull the drawer out by pulling one handle, unless the length of the drawer exceed μc.

48. If one cord of a sash-window break, find the coefficient of friction between the sash and the window-frame in order that the other weight may still support the window.

49. A circular hoop, of radius one foot, hangs on a horizontal bar and a man hangs by one hand from the hoop. If the coefficient of friction between the hoop and the bar be $\frac{1}{\sqrt{3}}$, find the shortest possible distance from the man's hand to the bar, the weight of the hoop being neglected.

50. An isosceles triangular lamina, whose vertical angle is 2β, rests in a vertical plane with its base along a line of greatest slope on a rough inclined plane whose inclination to the horizon is α. Shew that equilibrium will be maintained so long as β lies within the limits $\tan^{-1}\mu$ to $\frac{\pi}{3}$ inclusive.

If a horizontal force, applied to the vertex of the triangle and tending down the plane, be on the point of disturbing the equilibrium both by rolling and sliding at the same instant, shew that β is given by the equation $3\tan\beta = 2\mu + \mu\cos 2\alpha - \sin 2\alpha$.

51. Three equal circular discs, A, B, and C, are placed in contact with each other upon a smooth horizontal plane, and, in addition, B and C are in contact with a rough vertical wall. If the coefficient of friction between the circumferences of the discs and also between the discs and wall be $2-\sqrt{3}$, shew that no motion will ensue when A is pushed perpendicularly towards the wall with any force P.

52. If the centre of gravity of a wheel and axle be at a distance a from the axis, shew that the wheel can rest with the plane through the axis and the centre of gravity inclined at an angle less than θ with the vertical, given by $\sin\theta = \frac{b}{a}\sin\phi$, where ϕ is the angle of friction, and b is the radius of the axle.

53. A particle, of weight w, rests on a rough inclined plane, of weight W, whose base rests on a rough table, the coefficients of friction being the same. If a gradually increasing force be applied to the particle w, find whether it will move up the plane before the plane slides on the table, the angle of inclination of the plane being α.

54. A rough cylinder, of weight W', lies with its axis horizontal upon a plane, whose inclination to the horizontal is α, whilst a man, of weight W, (with his body vertical), stands upon the cylinder and keeps it at rest. If his feet be at A and a vertical section of the cylinder through A touch the plane at B, shew that the angle, θ, subtended by AB at the centre of the section, the friction being sufficient to prevent any sliding, is given by the equation

$$\frac{\sin(\theta+\alpha)}{\sin\alpha} = 1 + \frac{W'}{W}.$$

55. Two rough uniform spheres, of equal radii but unequal weights W_1 and W_2, rest in a spherical bowl, the line joining their centres being horizontal and subtending an angle 2α at the centre of the bowl; shew that the coefficient of friction between them is not less than

$$\frac{W_1 - W_2}{W_1 + W_2}\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

56. Two rigid weightless rods are firmly jointed, so as to be at right angles, a weight being fixed at their junction, and are placed over two rough pegs in the same horizontal plane, whose coefficients of friction are μ and μ'. Shew that they can be turned either way from their symmetrical position through an angle $\frac{1}{2}\tan^{-1}\frac{\mu+\mu'}{2}$, without slipping.

57. A sphere, of weight W, is placed on a rough plane, inclined to the horizon at an angle α, which is less than the angle of friction; shew that a weight $W\dfrac{\sin\alpha}{\cos\alpha-\sin\alpha}$, fastened to the sphere at the upper end of the diameter which is parallel to the plane, will just prevent the sphere from rolling down the plane.

What will be the effect of slightly decreasing or slightly increasing this weight?

58. Two equal uniform rods are joined rigidly together at one extremity of each to form a V, with the angle at the vertex 2α, and are placed astride a rough vertical circle of such a radius that the centre of gravity of the V is in the circumference of the circle, the angle of friction being ϵ. Shew that, if the V be just on the point of motion when the line joining its vertex with the centre of the circle is horizontal, then $\sin\epsilon=\sqrt{\sin\alpha}$.

If the rods be connected by a hinge and not rigidly connected and the free ends be joined by a string, shew that the joining string will not meet the circle if $\sin\alpha$ be $>\frac{1}{2}$; if this condition be satisfied, shew that if the V is just on the point of slipping when the line joining its vertex to the centre is horizontal, the tension of the string will be $\frac{W}{2}\sqrt{1+\operatorname{cosec}\alpha}$, where W is the weight of either rod.

59. A rectangular beam, of weight W, is constrained by guides to move only in its own direction, the lower end resting on a smooth floor. If a smooth inclined plane of given slope be pushed under it by a horizontal force acting at the back of the inclined plane, find the force required.

If there be friction between the floor and the inclined plane, but nowhere else, what must be the least value of μ so that the inclined plane may remain, when left in a given position under the beam, without being forced out.

60. Two weights, P and Q, hang at the ends of a light string passing over a fixed pulley of radius a which is capable of turning about a rough axle, of radius b; if P be on the point of overcoming Q, shew that it equals $Q\dfrac{b\sin\lambda+a}{b\sin\lambda-a}$, where λ is the angle of friction.

61. In a wheel and axle, if the axle rest on rough bearings, shew that the least power which, acting downwards, will raise a weight W is

$\dfrac{b(1+\sin\lambda)}{a-b\sin\lambda}W$, where a and b are the radii of the wheel and axle, and λ is the coefficient of friction.

62. Two small rings, each of weight W, slide one upon each of two rods in a vertical plane, each inclined at an angle α to the vertical; the rings are connected by a fine elastic string of natural length $2a$, and whose modulus of elasticity is λ; the coefficient of friction for each rod and ring is $\tan\beta$; shew that, if the string be horizontal, each ring will rest at any point of a segment of the rod whose length is

$$W\lambda^{-1}a\operatorname{cosec}\alpha\,\{\cot(\alpha-\beta)-\cot(\alpha+\beta)\}.$$

63. A wedge, with angle 60°, is placed upon a smooth table, and a weight of 20 lbs. on the slant face is supported by a string lying on that face which, after passing through a smooth ring at the top, supports a weight W hanging vertically; find the magnitude of W.

Find also the horizontal force necessary to keep the wedge at rest

(1) when the ring is not attached to the wedge,

(2) when it is so attached.

Solve the same question supposing the slant face of the wedge to be rough, the coefficient of friction being $\dfrac{1}{\sqrt{3}}$ and the 20 lb. weight on the point of moving down.

64. Shew that the power necessary to move a cylinder, of radius r and weight W, up a plane inclined at an angle α to the horizon by means of a crowbar of length l inclined at an angle β to the horizon is

$$\frac{Wr}{l}\,\frac{\sin\alpha}{1+\cos(\alpha+\beta)}.$$

65. A letter-weigher consists of a uniform plate in the form of a right-angled isosceles triangle ABC, of mass 3 ozs., which is suspended by its right angle C from a fixed point to which a plumb-line is also attached. The letters are suspended from the angle A, and their weight read off by observing where the plumb-line intersects a scale engraved along AB, the divisions of which are marked 1 oz., 2 oz., 3 oz., etc. Shew that the distances from A of the divisions of the scale form a harmonic progression.

66. A ladder, of length l feet and weight W lbs., and uniform in every respect throughout, is raised by two men A and B from a horizontal to a vertical position. A stands at one end and B, getting underneath the ladder, walks from the other end towards A holding successive points of the ladder above his head, at the height of d feet from the ground, the force he exerts being vertical. Find the force exerted by B when thus supporting a point n feet from A, and shew that the work done by him in passing from the n^{th} to the $(n-1)^{\text{th}}$ foot is $\dfrac{Wld}{2n(n-1)}$.

When must A press his feet *downwards* against his end of the ladder?

ANSWERS TO THE EXAMPLES.

I. (PAGE 13.)

1. (i) 25; (ii) $3\sqrt{3}$; (iii) 13; (iv) $\sqrt{61}$; (v) 60°; (vi) $\sqrt{505}$; (vii) 3.

2. 20 lbs. wt.; 4 lbs. wt.

3. $\sqrt{2}$ lbs. wt. in a direction south-west. **4.** 205 lbs. wt.

5. P lbs. wt. at right angles to the first component.

6. 2 lbs. wt. **7.** 20 lbs. wt. **8.** 17 lbs. wt.

9. 60°. **10.** 3 lbs. wt.; 1 lb. wt.

11. (i) 120°; (ii) $\cos^{-1}\left(-\frac{7}{8}\right)$, *i.e.* 151° 3′.

12. $\cos^{-1}\left(-\frac{1}{2}\frac{A^2+B^2}{A^2-B^2}\right)$.

13. In the direction of the resultant of the two given forces.

II. (PAGE 17.)

1. $5\sqrt{3}$ and 5 lbs. wt. **2.** $\frac{1}{2}P\sqrt{2}$; $\frac{12}{13}P$. **3.** 50 lbs. wt.

4. Each is $\frac{1}{3}100\sqrt{3}$, *i.e.* 57·736, lbs. wt.

5. 36·604 and 44·84 lbs. wt. **6.** $P(\sqrt{3}-1)$ and $\frac{P}{2}(\sqrt{6}-\sqrt{2})$.

8. $F\sqrt{3}$ and $2F$. **9.** $F\sqrt{2}$ at 135° with the other component.

10. $10\sqrt{5}$ at $\tan^{-1}\frac{1}{2}$ (*i.e.* 22·36 at 26° 34′) with the vertical.

III. (PAGE 23.)

1. $1:1:\sqrt{3}$. **2.** $\sqrt{3}:1:2$. **3.** 120°.

4. 90°, 156° 23′, 113° 37′.

IV. (PAGES 28—30.)

1. 40. **2.** $\cos^{-1}\left(-\frac{1}{4}\right)$, *i.e.* 104° 29′.

3. $2\sqrt{3}$ and $\sqrt{3}$ lbs. wt. **4.** $15\sqrt{3}$ and 15 lbs. wt.

5. 5 : 4. **6.** 5 and 13. **9.** 12 lbs. wt.

16. The straight line passes through C and the middle point of AB.

19. The required point bisects the line joining the middle points of the diagonals.

20. Through B draw BL, parallel to AC, to meet CD in L; bisect DL in X; the resultant is a force through X, parallel to AD, and equal to twice AD.

V. (PAGE 35.)

1. 4 lbs. wt. in the direction AQ.

2. $\sqrt{50+32\sqrt{2}}$ at an angle $\tan^{-1}\frac{7+4\sqrt{2}}{17}$, *i.e.* 9·76 lbs. wt. at 36° 40′, with the first force.

3. $2P$ in the direction of the middle force.

4. $7P$ at $\cos^{-1}\frac{11}{14}$, *i.e.* 38° 13′, with the third force.

5. $\sqrt{3}P$ at 30° with the third force.

6. 12·31 making an angle $\tan^{-1} 5$, *i.e.* 78° 41′, with AB.

7. 14·24 lbs. wt. **8.** 5 lbs. wt. opposite the third force.

9. $\frac{1}{4}P(\sqrt{5}+1)\sqrt{10+2\sqrt{5}}$ bisecting the angle between Q and R.

10. 10 lbs. wt. towards the opposite angular point.

11. $\sqrt{125+68\sqrt{3}}$ lbs. wt. at an angle $\tan^{-1}\frac{64+19\sqrt{3}}{23}$, *i.e.* 15·61 lbs. wt. at 76° 39′, with the first force.

13. $P \times 5{\cdot}027$.

14. (i) 17·79 lbs. wt. at 66° 29′ with the fixed line;
(ii) 10·19 lbs. wt. at 36° 9′ with the fixed line;
(iii) 39·50 lbs. wt. at 111° 46′ with the fixed line.

VI. (PAGES 39—41.)

1. $\frac{W}{2}(\sqrt{6}-\sqrt{2})$; $W(\sqrt{3}-1)$. **2.** $2\frac{2}{3}$ and $3\frac{1}{3}$ lbs. wt.

3. 126 and 32 lbs. wt. **4.** 56 and 42 lbs. wt.

5. 48 and 36 lbs. wt. **6.** 4, 8, and 12 lbs. wt.

7. W. **8.** 120 lbs. wt.

9. The inclined portions of the string make 60° with the vertical and the pressure is $W\sqrt{3}$.

10. 7·23 lbs. wt. **11.** The weights are equal.

12. 1·34 inches. **13.** $2\frac{4}{5}$ and $9\frac{3}{5}$ lbs. wt.

14. 14 lbs. wt. **15.** 6 ft. 5 ins.; 2 ft. 4 ins.

16. They are each equal to the weight of the body.

18. $2P$. **20.** $\frac{W}{2}\sec\frac{C}{2}$. **22.** W; $W\sqrt{2}$.

VII. (PAGES 47—49.)

1. (i) $R=11$, $AC=7$ ins.; (ii) $R=30$, $AC=1$ ft. 7 ins.;
(iii) $R=10$, $AC=1$ ft. 6 ins.

2. (i) $R=8$, $AC=25$ ins.; (ii) $R=8$, $AC=-75$ ins.;
(iii) $R=17$, $AC=-19\frac{1}{17}$ ins.

3. (i) $Q=9$, $AB=8\frac{1}{2}$ ins.; (ii) $P=2\frac{3}{4}$, $R=13\frac{3}{4}$;
(iii) $Q=6\frac{3}{4}$, $R=12\frac{3}{4}$.

4. (i) $Q=25$, $AB=3\frac{3}{50}$ ins.; (ii) $P=29\frac{3}{4}$, $R=18\frac{3}{4}$;
(iii) $Q=2\frac{4}{7}$, $R=3\frac{3}{7}$.

5. 15 and 5 lbs. wt. **6.** $43\frac{1}{3}$ and $13\frac{1}{3}$ lbs. wt.

8. 98 and 70 lbs. wt.

9. The block must be 2 ft. from the stronger man.

10. 4 ft. 3 ins. **11.** 1 lb. **12.** 1 foot.

13. 20 lbs.; 4 ins.; 8 ins. **14.** $14\frac{3}{8}$ ins.; $10\frac{5}{8}$ ins.

16. 40 and 35 lbs. wt. **17.** $\frac{1}{2}W$.

18. The force varies inversely as the distance between his hand and his shoulders.

19. (i) 100 and 150 lbs. wt.; (ii) 50 and 100 lbs. wt.; (iii) 25 and 75 lbs. wt.

20. 1 lb. wt. at 5 ft. from the first.

VIII. (PAGES 61—63.)

1. 10·1. **2.** $5\sqrt{3}$ ft.-lbs. **3.** $75\sqrt{3}=129·9$ ft.-lbs.

4. 3 ft. 8 ins. from the 6 lb. wt.

5. At a point distant 6·6 feet from the 20 lbs.

6. $2\frac{6}{7}$ ft. from the end. **7.** $2\frac{2}{3}$ lbs.

8. $2\frac{1}{3}$ lbs. **9.** (i) 4 tons each; (ii) $4\frac{1}{3}$ tons, $3\frac{2}{3}$ tons.

10. B is 3 inches from the peg. **11.** $\frac{6}{7}$ cwt.

12. One-quarter of the length of the beam. **13.** 55 lbs. wt.

14. The weight is $3\frac{1}{2}$ lbs. and the point is $7\frac{1}{2}$ ins. from the 5 lb. wt.

15. 3 ozs. **16.** $85\frac{1}{2}$, $85\frac{1}{2}$, and 29 lbs. wt.

17. $87\frac{1}{6}$, $87\frac{1}{6}$, and $53\frac{2}{3}$ lbs. wt. **18.** $1\frac{17}{28}$ ins. from the axle.

19. $2\sqrt{2}$ lbs. wt., parallel to CA, and cutting AD at P, where AP equals $\frac{2}{3}AD$.

20. $2P$ acting along DC.

21. The resultant is parallel to AC and cuts AD at P, where AP is $\frac{2}{5}$ ft.

22. $20\sqrt{5}$ lbs. wt. cutting AB and AD in points distant from A 8 ft. and 16 ft. respectively.

23. $P\sqrt{3}$ perpendicular to BC and cutting it at Q where BQ is $\frac{3}{2}BC$.

29. The required height is $\frac{1}{2}l\sqrt{2}$, where l is the length of the rope.

31. A straight line dividing the exterior angle between the two forces into two angles the ratio of whose sines is equal to the ratio of the forces.

IX. (PAGE 68.)

2. 9 ft.-lbs. **3.** 6.

4. A force equal, parallel, and opposite, to the force at C, and acting at a point C' in AC, such that CC' is $\frac{2}{5}AB$.

X. (PAGES 81—84.)

2. 45°. **3.** $10\sqrt{2}$ and 10 lbs. wt.

4. The length of the string is AC. **5.** $\frac{1}{3}W\sqrt{3}$; $\frac{2}{3}W\sqrt{3}$.

8. $\frac{a}{l}$ must be <1 and $>\frac{1}{2}$. **12.** $\frac{3}{5}\sqrt{5}$ and $\frac{6}{5}\sqrt{5}$ lbs. wt.

14. $W \operatorname{cosec} \alpha$ and $W \cot \alpha$. **15.** $\frac{1}{3}W\sqrt{3}$.

16. 30°; $\frac{2}{3}W\sqrt{3}$; $W\sqrt{3}$. **17.** $\sqrt{7} : 2\sqrt{6}$.

18. $\frac{40}{3}\sqrt{3}$ lbs. wt. **19.** $6\frac{2}{3}$ lbs. wt.

21. $h\sqrt{h^2+a^2\sin^2\alpha}/(h+a\cos\alpha)$, where $2a$ is the height of the picture.

XI. (PAGES 94—97.)

1. $\frac{1}{4}W\sqrt{3}$. **2.** $\frac{1}{6}W\sqrt{3}$. **4.** $\frac{1}{2}W\cot\alpha$; $\frac{5}{6}W\cot\alpha$.

6. $\dfrac{W\sin\beta}{\sin(\alpha+\beta)}$; $\dfrac{W\sin\alpha}{\sin(\alpha+\beta)}$; $\tan^{-1}\left(\dfrac{\cot\beta-\cot\alpha}{2}\right)$.

8. $\frac{3}{8}$ lb. wt. **11.** $AC=a$; the tension $=2W\sqrt{3}$.

14. The pressures at the edge and the base are respectively 3·24 and 4·814 ozs. wt.

15. $W . r/2\sqrt{R^2 - r^2}$. **18.** $W\dfrac{a}{b}$. **23.** $\frac{1}{8}W\sqrt{6}$.

24. $133\frac{1}{3}$ and $166\frac{2}{3}$ lbs. wt. **26.** $17\frac{11}{36}$ and $\frac{25}{36}$ lbs. wt.

27. 20 lbs. wt.

XII. (PAGE 100.)

1. The force is $4\sqrt{2}$ lbs. wt. inclined at 45° to the third force, and the moment of the couple is $20a$, where a is the side of the square.

2. The force is $5P\sqrt{2}$, parallel to DB, and the moment of the couple is $21Pa$, where a is the side of the square.

3. The force is 6 lbs. wt., parallel to CB, and the moment of the couple is $\dfrac{21\sqrt{3}a}{2}$.

XIII. (PAGES 104, 105.)

1. The side makes an angle $\tan^{-1}(5)$ with the horizon.

2. $15a$. **3.** $(n+2)\sqrt{b^2+c^2}$.

4. The weight of the table. **6.** 10 lbs.

7. On the line joining the centre to the leg which is opposite the missing leg, and at a distance from the centre equal to one-third of the side of the square.

8. 120 lbs. **9.** $\sin^{-1}\dfrac{p}{p+w}$.

11. The pressure on A is $W\dfrac{\cos A}{2\sin B\sin C}$.

XIV. (PAGES 115, 116.)

1. $1\frac{1}{3}$, $1\frac{3}{5}$, and $1\frac{3}{5}$ feet. **2.** 2, $2\frac{2}{3}$, and $1\frac{3}{5}$ feet.

3. $2\sqrt{5}$, 3, and 3 feet.

6. The pressure at the point A of the triangle is $\dfrac{w}{3} + W\dfrac{a}{c\sin B}$, where a is the distance of the weight W from the side BC.

10. 60°. **12.** $\cos^{-1}\frac{7}{25}$, *i.e.* 73° 44′.

XV. (PAGES 119, 120.)

1. $4\frac{2}{3}$ inches from the end. **2.** 15 inches from the end.

3. $2\frac{5}{9}$ feet. **4.** $\frac{20}{21}$ inch from the middle.

5. $7\frac{1}{3}$ inch from the first particle.

6. It divides the distance between the two extreme weights in the ratio of 7 : 2.

7. 5 : 1. **8.** 1·335... feet. **9.** $\dfrac{2n}{3}$ inches.

10. 12 lbs.; the middle point of the rod.

XVI. (PAGES 124, 125.)

1. One-fifth of the side of the square.

2. $\frac{3a}{4}$ from AB; $\frac{a}{4}$ from AD.

3. At a point whose distances from AB and AD are 16 and 15 inches respectively.

4. $7\frac{1}{3}$ and $8\frac{1}{3}$ inches. 5. $\frac{a}{6}\sqrt{19}$; $\frac{a}{30}\sqrt{283}$.

7. At the centre of gravity of the lamina.

8. $8\frac{1}{3}$ and $11\frac{1}{3}$ inches. 10. 2 : 1 : 1.

12. At a point whose distances from BC and CA are respectively $\frac{3}{13}$ths and $\frac{4}{13}$ths of the distances of A and B from the same two lines.

14. It divides the line joining the centre to the fifth weight in the ratio of 5 : 9.

18. One quarter of the side of the square. 20. $4\frac{1}{3}$ inches from A.

21. It passes through the centre of the circle inscribed in the triangle.

XVII. (PAGES 129—131.)

1. $2\frac{1}{24}$ inches from the joint.

2. $5\frac{7}{8}$ inches from the lower end of the figure.

3. It divides the beam in the ratio of 5 : 11.

4. At the centre of the base of the triangle. 5. $7\frac{3}{8}$ inches.

7. One inch from the centre of the larger sphere.

8. Its distance from the centre of the parallelogram is one-ninth of a side.

9. The distance from the centre is one-twelfth of the diagonal.

10. The distance from the centre is $\frac{1}{21}$th of the diagonal of the square.

11. It divides the line joining the middle points of the opposite parallel sides in the ratio of 5 : 7.

12. $\frac{1}{3}\sqrt{3}$ inches from O. 14. $\frac{4a}{3}$.

15. A' bisects AD, where D is the middle point of BC.

16. It divides GA in the ratio $\sqrt{m} - 1 : m\sqrt{m} - 3\sqrt{m} + 1$.

17. The height of the triangle is $\frac{3-\sqrt{3}}{2}$ side of the square.

18. $\frac{15}{194}$ inches from the centre.

19. The centre of the hole must be 16 inches from the centre of the disc.

20. It is at a distance $\frac{b^3}{a^2+ab+b^2}$ from the centre of the larger sphere.

21. $\frac{45}{56}h$, where h is the height of the cone. **22.** 13·532 inches.

23. The height, x, of the part scooped out is given by the equation

$$x^3+hx^2+h^2x=\frac{h^3}{3}.$$

XVIII. (PAGES 137—139.)

1. By 7, 8, and 9 lbs. wt. respectively. **2.** $1\frac{7}{10}$ inch.

6. 4 : 5. **10.** At the centre of gravity of the triangle.

14. $2\sin^{-1}\frac{1}{3}$. **15.** $\sqrt{6}:1$.

16. The height of the cone must be to the height of the cylinder as $7-\sqrt{43}:1$.

19. It divides the axis of the original cone in the ratio 3 : 5.

XIX. (PAGES 150—152.)

1. $6\frac{2}{3}$ inches. **2.** $\frac{6}{\sqrt[3]{\pi}}$ inches. **4.** $\frac{W}{6}$.

7. 120; $\frac{1}{60}$th.

8. 18 if they overlap in the direction of their lengths, and 8 if in the direction of their breadths.

11. $\sqrt{3}$ times the radius of the hemisphere. **12.** $1:\sqrt{2}$. **14.** $4r$.

18. The string makes an angle $\cos^{-1}\left(\frac{W\sin\alpha}{P}\right)$ with the plane; the equilibrium is stable.

19. The line from the fixed point to the centre is inclined at an angle $\sin^{-1}\left[\frac{W-w}{P+W+w}\,\frac{r}{c}\right]$ to the vertical; the equilibrium is stable.

XX. (PAGES 159—161.)

1. 5 feet. **2.** 4 feet from the first weight; toward the first weight. **3.** 11 : 9. **4.** 2 lbs. **6.** 4 lbs.

7. $9\frac{4}{5}$ lbs. **8.** 6 ins. from the 27 ounces; $1\frac{5}{7}$ inch.

9. 1 foot. **10.** 360 stone wt. **11.** 21 lbs. wt.

12. 15 lbs. wt. **13.** $2\sqrt{2}$ at 45° to the lever. **14.** 50 lbs. wt.

15. The long arm makes an angle $\tan^{-1}\frac{7}{\sqrt{3}}$ with the horizon.

16. $8\frac{1}{2}$ feet. **19.** 20 lbs. **20.** The weight of $2\frac{1}{2}$ cwt.

21. $\frac{n}{6}(\sqrt{3}-1)\,a$.

XXI. (PAGES 166, 167.)

1. (i) 320; (ii) 7; (iii) 3. **2.** (i) 7; (ii) $45\frac{1}{2}$; (iii) 7; (iv) 6.

3. 290 lbs. **4.** $10\frac{3}{4}$ lbs. **5.** 5 lbs. **7.** 5 lbs.

9. 49 lbs.; 1 lb. **10.** $4w$; $21w$. **12.** $9\frac{15}{17}$ lbs. wt.

13. 18 lbs. wt.

XXII. (PAGE 169.)

1. 6 lbs. **2.** 4 strings; 2 lbs.

3. 47 lbs.; 6 pulleys. **4.** 7 strings; 14 lbs.

5. $\frac{W}{n+1}$, where n is the number of strings. **6.** 9 stone wt.

7. The cable would support $2\frac{1}{4}$ tons.

XXIII. (PAGES 174, 175.)

1. (i) 30 lbs.; (ii) 4 lbs.; (iii) 4.

2. (i) 161 lbs. wt.; (ii) 16 lbs. wt.; (iii) $\frac{1}{2}$ lb.; (iv) 5.

3. 10 lbs. wt.; the point required divides the distance between the first two strings in the ratio of 23 : 5.

4. $\frac{11}{15}$ inch from the end. **5.** $18\frac{6}{53}$.

6. $\frac{4}{7}$ inch from the end. **8.** $W=7P+4w$; 8 ozs.; 1 lb. wt.

9. 4; 1050 lbs. **10.** 4.

12. $W=P(2^n-1)+W'(2^{n-1}-1)$.

XXIV. (Pages 180, 181.)

1. 12 lbs.; 20 lbs. **2.** 30°; $W\frac{\sqrt{3}}{2}$.

3. 103·92 lbs. wt. **5.** 3 : 4; $2P$. **6.** $\sqrt{3} : 1$.

7. $\cos^{-1}\frac{11}{16}$; $\sin^{-1}\frac{1}{4}$ to the plane. **8.** $\frac{1}{\sqrt{3}}$ lbs. wt.; $\frac{7}{\sqrt{3}}$ lbs. wt.

9. 6 lbs. wt. **11.** $16\frac{1}{2}$ lbs. **12.** $\frac{\sin\alpha}{\sin\beta - \sin\alpha}$ tons.

14. The point divides the string in the ratio $1 : \sin\alpha$.

16. 17·374 lbs. wt.; 46·884 lbs. wt.

17. 10·318 lbs. wt.; 12·208 lbs. wt.

18. 16·12 lbs. wt.; 34·056 lbs. wt.

XXV. (Pages 185, 186.)

1. 7 lbs. wt. **2.** 120 lbs. wt.; 140 lbs. wt.; $110\frac{10}{11}$ lbs. wt.

3. 20 inches. **4.** 7 feet. **5.** $3\frac{3}{5}$ tons.

6. 3 lbs. wt. **7.** 55 lbs. **8.** $23\frac{1}{3}$ lbs. wt.

9. $2\frac{1}{3}$ lbs. wt. **10.** 360 lbs. **11.** 120 lbs.

XXVI. (Pages 193—195.)

1. 11 lbs. **2.** $26\frac{1}{4}$ lbs. **3.** 2 ozs.

4. 2 : 3; 6 lbs. **5.** 24·494 lbs. **6.** $10 : 2\sqrt{26}$.

7. $\frac{6}{5}\sqrt{110}$ inches; $\sqrt{110}$ lbs. **9.** 2*s.* 3*d.*; 1*s.* $9\frac{1}{3}$*d.*

10. He will lose one shilling. **12.** ·995; 1·005.

13. $\frac{P-Q}{2}$; $\frac{P+Q}{2}$. **14.** $\frac{w-P}{P-w'}$; $\frac{ww'-P^2}{P-w'}$.

15. $R - \frac{(Q-R)^2}{P-Q}$. **16.** 16 lbs.

XXVII. (Pages 199, 200.)

1. $34\frac{5}{6}$ inches to the right of the fulcrum.

2. 2 inches from the end; 1 inch. **3.** 32 inches.

4. $\frac{2}{3}$ inch; $4\frac{1}{2}$ lbs. **5.** 4 inches.

6. 16 lbs.; 8 inches beyond the point of attachment of the weight.

7. $\frac{2}{9}$ lb.; $\frac{18}{19}$ lb. **8.** 26 lbs.; 14 lbs.; 10 inches from the fulcrum.

9. 3 lbs. **10.** $15\frac{7}{8}$ lbs.; $6\frac{7}{8}$ lbs.; 4 inches.

11. It is 10 inches from the point at which the weight is attached.

12. 3 ozs. **13.** 30 inches.

15. The numbers marked on the machine must each be increased by $\frac{1}{6}$ lb.

16. The numbers marked on the machine must each be increased by $\frac{x}{y}\frac{W}{10}$, where x and y are respectively the distances of the centre of gravity of the machine from the end and the fulcrum, and W is the weight of the machine.

17. He cheats his customers, or himself, according as he decreases, or increases, the movable weight.

XXVIII. (PAGE 207.)

(*In the following examples* π *is taken to be* $\frac{22}{7}$).

1. 4400 lbs. 2. $5\frac{8}{11}$ inches. 3. $\frac{25}{27}$ lbs. wt.
4. $1\frac{69}{176}$ lbs. wt. 5. $4\frac{84}{99}$ lbs. wt. 6. $13\frac{23}{45}$ tons wt.
7. $6\frac{2}{7}$ tons wt. 8. $50\frac{10}{11}$ lbs. wt. 9. $4\frac{92}{175}$ inches.
10. $4525\frac{5}{7}$. 11. $5430\frac{6}{7}$.

XXIX. (PAGES 216, 217.)

1. 10 lbs. wt.; $12\frac{1}{2}$ lbs. wt.; $10\sqrt{17}$ and $\frac{15}{2}\sqrt{17}$ lbs. wt. respectively, inclined at an angle $\tan^{-1} 4$ with the horizontal.

2. $\frac{P}{W}=\frac{\sqrt{2}}{3}=\cdot 4714.$

3. $10\sqrt{10}$ lbs. wt. at an angle $\tan^{-1} 3$ with the horizon.

6. $\frac{1}{4}$. 7. $\frac{4}{3}\sqrt{3}$ lbs. wt. 9. $\frac{\sqrt{3}}{15}$.

10. $\sin\beta=\sin\alpha+\mu\cos\alpha$. 11. $\tan^{-1}\left(\mu\frac{W_1+W_2}{W_1-W_2}\right)$.

14. $a\times\cdot 134$. 15. At an angle equal to the angle of friction.

XXX. (PAGES 221, 222.)

1. $11\frac{2}{3}$ lbs. wt. 2. 45°.
5. It can be ascended as far as the centre. 8. 50 feet.

9. $w\frac{\tan\alpha-2\mu}{\mu-\tan\alpha}$.

XXXI. (PAGES 223, 224.)

1. $\tan^{-1}\frac{1}{2}$; height = twice diameter. 3. $\frac{a}{\mu}$.

4. 45°. 6. $2\tan^{-1}\frac{\sqrt{3}}{12}$. 8. Unity.

XXXII. (PAGES 230—234.)

11. $\frac{\sqrt{3}}{30}$. 15. $\mu\left(\frac{W}{w}+1\right)\sqrt{b^2-a^2}$. 18. $\frac{7}{8}$.

20. The required force is $\frac{3}{5}W$ at an angle $\cos^{-1}\frac{11}{14}$ with the line of greatest slope.

.21. In a direction making an angle $\cos^{-1} 5\frac{\sqrt{3}}{9}$ with the line of greatest slope.

24. If $\mu \cot \alpha$ be greater than unity, there is no *limiting* position of equilibrium, *i.e.*, the particle will rest in any position.

28. 60°.

XXXIII. (PAGES 244, 245.)

1. 3808 ft.-lbs. **2.** 7,392,000 ft.-lbs.; $7\frac{7}{15}$ H.P.
3. 21120. **4.** 23,040,000 ft.-lbs.; $5\frac{9}{11}$ H.P.
5. 1000 feet. **6.** $9\frac{27}{32}$ hours. **7.** $8\frac{13}{14}$.
8. 44·352. **9.** 660,000 ft.-lbs.; 30 H.P.
10. 7766. **11.** 1500 ft.-lbs.
12. 47040 ft.-lbs.; 4 cwt.; 105 feet.
13. 6272 ft.-lbs. **14.** $4\frac{1}{8}$ ft.-lbs.

XXXIV. (PAGES 248—250.)

1. At P where BP equals $\frac{1}{3}BC$; $\frac{3W}{2}$.

2. At A, the action is $10\sqrt{19}$ lbs. wt. at $\tan^{-1}\frac{4}{\sqrt{3}}$ with the horizon; at B, $10\sqrt{3}$ lbs. wt. horizontally; at C, $10\sqrt{7}$ lbs. wt. at $\tan^{-1}\frac{2}{\sqrt{3}}$ with the horizon. **4.** $\frac{W}{2}\tan\frac{ACB}{2}$.

7. $\frac{3W}{\sqrt{5}}$; $\frac{W}{5}\sqrt{10}$ at $\tan^{-1}\frac{1}{3}$ to the horizontal.

8. $\frac{W}{2}$; $\frac{3W}{2}$; $\frac{\sqrt{3}W}{4}$. **10.** Half the weight of the middle rod.

12. One-eighth of the total weight of the rods, acting in a horizontal direction.

XXXVI. (PAGES 261, 262.)

1. 42·9 lbs. wt.; 19·91 lbs. wt. **2.** $1\frac{1}{3}$ and $1\frac{2}{3}$ tons wt.
3. 39 lbs. wt.; 25·8 lbs. wt. at 1° 40′ to the horizon.
4. 152·3 and 267·96 lbs. wt. **5.** ·41.
6. 124°, 103°, 133°. **7.** 26·9 lbs. wt.
8. 74 lbs. wt.; 12·7 lbs. wt. **9.** 37·8 and 85·1 lbs. wt.

10. 15·2 lbs. **11.** Each equals the wt. of 10 cwt.

12. ·46; 91·2 and 57·2 lbs. wt.

13. 3·4, 6·6, 3·67, 7·55, and 5 cwt. respectively.

14. 58·1, 65·8, 37·4, 33·2, and 29 lbs. wt. respectively.

MISCELLANEOUS EXAMPLES. (PAGES 263—272.)

5. The centre of the inscribed circle.

6. P divides AD in the ratio $1 : \sqrt{3}$.

21. $\frac{2}{5}$ lb. wt. **28.** $\frac{l}{2}$.

31. $W\frac{x\sqrt{a^2+b^2}}{ab}$, where a and b are the lengths of the sides of the frame and AP is x.

39. At a point distant $\sqrt{c^2+a^2}-a$ from the centre of the rod, where $2c$ is the length of the rod, and a is the distance from the centre of the given point.

44. The difference between the angles of inclination of the planes to the horizon must be not greater than the sum of the angles of friction.

45. $W\cos\lambda\sin(\alpha-\lambda)\operatorname{cosec}\alpha$, and $W\cos\lambda\sin\lambda\operatorname{cosec}\alpha$, where W is the weight of the rod.

48. One half the ratio of the depth to the width of the sash.

49. $\sqrt{3}$ feet. **53.** The particle will move first.

57. The equilibrium will be broken, or not, according as the weight is increased, or diminished.

59. $W\tan i$; $\frac{W}{W+W'}\tan i$, where W' is the weight of the inclined plane.

63. $W=10\sqrt{3}$; the force is (i) $5\sqrt{3}$ lbs. weight, and (ii) zero.

$W=\frac{20\sqrt{3}}{3}$; the force is (i) $\frac{10}{\sqrt{3}}$ lbs. wt., and (ii) zero.

65. He must press downwards when B has raised more than half the ladder.

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